回転球座標系における電磁流体力学の方程式

本稿では，地球ダイナモ・シミュレーションに必要となる回転球座標系$\left(\hat{r},\hat{\theta},\hat{\phi}\right)$における電磁流体力学の基礎方程式をまとめておく．特に，静止座標系と回転座標系における違いに焦点を当てる．図１に示すように，単位ベクトルは，半径方向に$\vec{e}_r$，ポロイダル方向に$\vec{e}_\theta$，トロイダル方向に$\vec{e}_\phi$，軸方向に$\vec{e}_z$とする．今，球座標系における回転軸は，北極と南極を貫くものであるとする．角速度ベクトルは，角速度Ωを用いて，次式で表される．

$$\vec{\Omega} = \Omega\vec{e}_z = \Omega\left(\vec{e}_r\cos\theta - \vec{e}_\theta\sin\theta\right) \tag{1}$$

静止座標系と回転座標系を区別するために，ここでは，$\hat{\ }$は回転座標系からみた変数に付与するものと約束する．ただし，表１に示すように，空間微分は座標系によらず不変であることから，$\left(\hat{r},\hat{\theta},\hat{\phi}\right)$には，$\hat{\ }$は以後は特に断らない限りは付与しないとする．

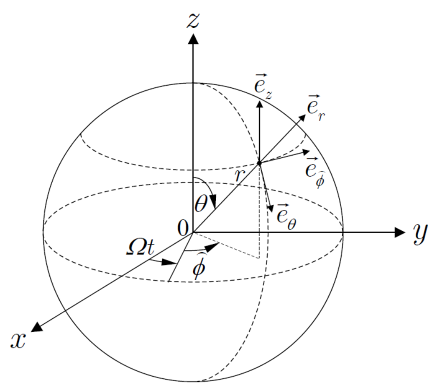


図１　回転球座標系における単位ベクトルの定義

表１　変数の座標系依存性

座標系で変わるもの	座標系で変わらないもの
速度（周方向成分），時間微分，スカラポテンシャル，電場，運動方程式	磁場，ベクトルポテンシャル，空間微分，電流密度（オームの法則），誘導方程式，エネルギー方程式

以下，式(2)から(5)は，回転座標系と慣性座標系の関係をまとめる．

＜回転角＞

$$\hat{\phi} = \phi - \Omega t \tag{2}$$

＜速度ベクトル＞

$$\hat{\vec{u}} = \vec{u} - \vec{\Omega}\times\vec{r} = \vec{u} - r\vec{\Omega}\times\vec{e}_r \quad \rightarrow \quad \begin{cases} \hat{u}_r = u_r \\ \hat{u}_\theta = u_\theta \\ \hat{u}_\phi = u_\phi - \Omega r \sin\theta \end{cases} \tag{3}$$

＜時間の偏導関数＞

$$\frac{\partial}{\partial \hat{t}} = \left[\frac{\partial}{\partial t} + \left(\vec{\Omega}\times\vec{r}\right)\cdot\vec{\nabla} - \vec{\Omega}\times\right] = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \phi} \tag{4}$$

＜スカラポテンシャル＞

$$\hat{\varphi}_e = \varphi_e - \left(\vec{\Omega}\times\vec{r}\right)\cdot\vec{a} = \varphi_e - \Omega r a_\phi \sin\theta \tag{5}$$

式(5)における a_ϕ は，磁場のベクトルポテンシャル $\vec{a}$ のトロイダル成分である．

静止座標系

＜質量保存式＞

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial\left(u_\theta \sin\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0 \tag{6}$$

＜運動方程式＞

$$\begin{aligned} &\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{u_\theta{}^2 + u_\phi{}^2}{r}\right) \\ &= -\frac{\partial p}{\partial r} + \mu\left(\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2} \\ &-\frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2u_\theta\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi} \end{aligned}\right) \\ &\quad + \frac{1}{\mu_m}\left(\frac{b_\theta}{r}\frac{\partial b_r}{\partial\theta} + \frac{b_\phi}{r\sin\theta}\frac{\partial b_r}{\partial\phi} - \frac{b_\theta{}^2 + b_\phi{}^2}{r} - b_\theta\frac{\partial b_\theta}{\partial r} - b_\phi\frac{\partial b_\phi}{\partial r}\right) + \rho g_r \end{aligned} \tag{7}$$

$$\begin{aligned} &\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{u_\theta u_r - u_\phi{}^2\cot\theta}{r}\right) \\ &= -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left(\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\theta}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\phi^2} \\ &+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi} \end{aligned}\right) \\ &\quad + \frac{1}{\mu_m}\left(b_r\frac{\partial b_\theta}{\partial r} + \frac{b_\phi}{r\sin\theta}\frac{\partial b_\theta}{\partial\phi} + \frac{b_r b_\theta - b_\phi{}^2\cot\theta}{r} - \frac{b_r}{r}\frac{\partial b_r}{\partial\theta} - \frac{b_\phi}{r}\frac{\partial b_\phi}{\partial\theta}\right) + \rho g_\theta \end{aligned} \tag{8}$$

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}+\frac{u_\phi u_r}{r}+\frac{u_\phi u_\theta\cot\theta}{r}\right)$$

$$=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}+\mu\left(\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}\\-\frac{u_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\end{array}\right) \tag{9}$$

$$+\frac{1}{\mu_m}\left(b_r\frac{\partial b_\phi}{\partial r}+\frac{b_\theta}{r}\frac{\partial b_\phi}{\partial\theta}+\frac{b_r b_\phi}{r}+\frac{b_\theta b_\phi\cot\theta}{r}-\frac{b_r}{r\sin\theta}\frac{\partial b_r}{\partial\phi}-\frac{b_\theta}{r\sin\theta}\frac{\partial b_\theta}{\partial\phi}\right)+\rho g_\phi$$

＜エネルギー方程式＞

$$\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial T}{\partial\phi}$$

$$=\alpha\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}\right\} \tag{10}$$

＜電荷保存則＞

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 j_r\right)+\frac{1}{r\sin\theta}\frac{\partial\left(j_\theta\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial j_\phi}{\partial\phi}=0 \tag{11}$$

＜Ohm の法則＞

$$j_r=\sigma\left(-\frac{\partial\varphi_e}{\partial r}-\frac{\partial a_r}{\partial t}+u_\theta b_\phi-u_\phi b_\theta\right) \tag{12}$$

$$j_\theta=\sigma\left(-\frac{1}{r}\frac{\partial\varphi_e}{\partial\theta}-\frac{\partial a_\theta}{\partial t}+u_\phi b_r-u_r b_\phi\right) \tag{13}$$

$$j_\phi=\sigma\left(-\frac{1}{r\sin\theta}\frac{\partial\varphi_e}{\partial\phi}-\frac{\partial a_\phi}{\partial t}+u_r b_\theta-u_\theta b_r\right) \tag{14}$$

＜誘導方程式（B 表示）＞

$$\frac{\partial b_r}{\partial t}+u_r\frac{\partial b_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_r}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial b_r}{\partial\phi}=b_r\frac{\partial u_r}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_r}{\partial\theta}+\frac{b_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi}$$

$$+\nu_m\left(\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial b_r}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial b_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 b_r}{\partial\phi^2}\\-\frac{2b_r}{r^2}-\frac{2}{r^2}\frac{\partial b_\theta}{\partial\theta}-\frac{2b_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial b_\phi}{\partial\phi}\end{array}\right) \tag{15}$$

$$\frac{\partial b_\theta}{\partial t}+u_r\frac{\partial b_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_\theta}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial b_\theta}{\partial \phi}+\frac{u_\theta b_r}{r}=b_r\frac{\partial u_\theta}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{b_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}+\frac{b_\theta u_r}{r}$$
$$+\nu_m\left(\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial b_\theta}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial b_\theta}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 b_\theta}{\partial\phi^2}\\ +\frac{2}{r^2}\frac{\partial b_r}{\partial\theta}-\frac{b_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial b_\phi}{\partial\phi}\end{array}\right) \tag{16}$$

$$\frac{\partial b_\phi}{\partial t}+u_r\frac{\partial b_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_\phi}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial b_\phi}{\partial \phi}+\frac{u_\phi b_r}{r}+\frac{u_\phi b_\theta\cot\theta}{r}=b_r\frac{\partial u_\phi}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{b_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}$$
$$+\frac{b_\phi u_r}{r}+\frac{b_\phi u_\theta\cot\theta}{r}+\nu_m\left(\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial b_\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial b_\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 b_\phi}{\partial\phi^2}\\ -\frac{b_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial b_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial b_\theta}{\partial\phi}\end{array}\right) \tag{17}$$

＜誘導方程式（A 表示；クーロンゲージ）＞

$$\frac{\partial a_r}{\partial t}+\frac{u_\theta}{r}\frac{\partial a_r}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial a_r}{\partial \phi}-\frac{u_\theta a_\theta}{r}-\frac{u_\phi a_\phi}{r}=-\frac{\partial\varphi_e}{\partial r}+u_\theta\frac{\partial a_\theta}{\partial r}+u_\phi\frac{\partial a_\phi}{\partial r}$$
$$+\nu_m\left(\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial a_r}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial a_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 a_r}{\partial\phi^2}\\ -\frac{2a_r}{r^2}-\frac{2}{r^2}\frac{\partial a_\theta}{\partial\theta}-\frac{2a_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial a_\phi}{\partial\phi}\end{array}\right) \tag{18}$$

$$\frac{\partial a_\theta}{\partial t}+u_r\frac{\partial a_\theta}{\partial r}+\frac{u_\phi}{r\sin\theta}\frac{\partial a_\theta}{\partial \phi}-\frac{u_\phi a_\phi\cot\theta}{r}=-\frac{1}{r}\frac{\partial\varphi_e}{\partial \theta}+u_r\left(\frac{1}{r}\frac{\partial a_r}{\partial\theta}-\frac{a_\theta}{r}\right)+u_\phi\left(\frac{1}{r}\frac{\partial a_\phi}{\partial\theta}\right)$$
$$+\nu_m\left(\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial a_\theta}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial a_\theta}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 a_\theta}{\partial\phi^2}\\ +\frac{2}{r^2}\frac{\partial a_r}{\partial\theta}-\frac{a_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial a_\phi}{\partial\phi}\end{array}\right) \tag{19}$$

$$\frac{\partial a_\phi}{\partial t}+u_r\frac{\partial a_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial a_\phi}{\partial \theta}=-\frac{1}{r\sin\theta}\frac{\partial\varphi_e}{\partial \phi}+u_r\left(\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial\phi}-\frac{a_\phi}{r}\right)+u_\theta\left(\frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial\phi}-\frac{a_\phi\cot\theta}{r}\right)$$
$$+\nu_m\left(\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial a_\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial a_\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 a_\phi}{\partial\phi^2}\\ -\frac{a_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial a_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial a_\theta}{\partial\phi}\end{array}\right) \tag{20}$$

＜磁荷不在の法則あるいはクーロンゲージ＞

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 b_r\right)+\frac{1}{r\sin\theta}\frac{\partial\left(b_\theta\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial b_\phi}{\partial\phi}=0 \tag{21}$$

クーロンゲージは，式(21)の b を a にすればよい.

回転座標系

慣性座標系からの違いを色で強調して示す.

＜質量保存式＞

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2u_r\right)+\frac{1}{r\sin\theta}\frac{\partial\left(u_\theta\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\hat{u}_\phi}{\partial\phi}=0 \tag{22}$$

＜運動方程式＞

$$\begin{aligned}
&\rho\left(\frac{\partial u_r}{\partial\hat{t}}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+\frac{\hat{u}_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\theta{}^2+\hat{u}_\phi{}^2}{r}\right)\underbrace{-\rho\Omega^2r\sin^2\theta}_{\text{centrifugal force}}\underbrace{-2\rho\Omega\hat{u}_\phi\sin\theta}_{\text{Coriolis force}}\\
&=-\frac{\partial p}{\partial r}+\mu\left(\begin{aligned}&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u_r}{\partial\phi^2}\\&-\frac{2u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2u_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial\hat{u}_\phi}{\partial\phi}\end{aligned}\right)\\
&\quad+\frac{1}{\mu_m}\left(\frac{b_\theta}{r}\frac{\partial b_r}{\partial\theta}+\frac{b_\phi}{r\sin\theta}\frac{\partial b_r}{\partial\phi}-\frac{b_\theta{}^2+b_\phi{}^2}{r}-b_\theta\frac{\partial b_\theta}{\partial r}-b_\phi\frac{\partial b_\phi}{\partial r}\right)+\rho g_r
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\rho\left(\frac{\partial u_\theta}{\partial\hat{t}}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\hat{u}_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}+\frac{u_\theta u_r-\hat{u}_\phi{}^2\cot\theta}{r}\right)\underbrace{-\rho\Omega^2r\sin\theta\cos\theta}_{\text{centrifugal force}}\underbrace{-2\rho\Omega\hat{u}_\phi\cos\theta}_{\text{Coriolis force}}\\
&=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\begin{aligned}&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\theta}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u_\theta}{\partial\phi^2}\\&+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial\hat{u}_\phi}{\partial\phi}\end{aligned}\right)\\
&\quad+\frac{1}{\mu_m}\left(b_r\frac{\partial b_\theta}{\partial r}+\frac{b_\phi}{r\sin\theta}\frac{\partial b_\theta}{\partial\phi}+\frac{b_rb_\theta-b_\phi{}^2\cot\theta}{r}-\frac{b_r}{r}\frac{\partial b_r}{\partial\theta}-\frac{b_\phi}{r}\frac{\partial b_\phi}{\partial\theta}\right)+\rho g_\theta
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\rho\left(\frac{\partial\hat{u}_\phi}{\partial\hat{t}}+u_r\frac{\partial\hat{u}_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial\hat{u}_\phi}{\partial\theta}+\frac{\hat{u}_\phi}{r\sin\theta}\frac{\partial\hat{u}_\phi}{\partial\phi}+\frac{\hat{u}_\phi u_r}{r}+\frac{\hat{u}_\phi u_\theta\cot\theta}{r}\right)\underbrace{+2\rho\Omega\left(u_r\sin\theta+u_\theta\cos\theta\right)}_{\text{Coriolis force}}\\
&=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}+\mu\left(\begin{aligned}&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\hat{u}_\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\hat{u}_\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\hat{u}_\phi}{\partial\phi^2}\\&-\frac{\hat{u}_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\end{aligned}\right)\\
&\quad+\frac{1}{\mu_m}\left(b_r\frac{\partial b_\phi}{\partial r}+\frac{b_\theta}{r}\frac{\partial b_\phi}{\partial\theta}+\frac{b_rb_\phi}{r}+\frac{b_\theta b_\phi\cot\theta}{r}-\frac{b_r}{r\sin\theta}\frac{\partial b_r}{\partial\phi}-\frac{b_\theta}{r\sin\theta}\frac{\partial b_\theta}{\partial\phi}\right)+\rho g_\phi
\end{aligned} \tag{25}$$

＜エネルギー方程式＞

$$\begin{aligned}
&\frac{\partial T}{\partial\hat{t}}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial\theta}+\frac{\hat{u}_\phi}{r\sin\theta}\frac{\partial T}{\partial\phi}\\
&=\alpha\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2T}{\partial\phi^2}\right\}
\end{aligned} \tag{26}$$

＜電荷保存則＞

式(11)に同じ

＜Ohm の法則＞

$$j_r = \sigma\left(-\frac{\partial \hat{\varphi}_e}{\partial r} - \frac{\partial a_r}{\partial \hat{t}} + u_\theta b_\phi - \hat{u}_\phi b_\theta \right) \tag{27}$$

$$j_\theta = \sigma\left(-\frac{1}{r}\frac{\partial \hat{\varphi}_e}{\partial \theta} - \frac{\partial a_\theta}{\partial \hat{t}} + \hat{u}_\phi b_r - u_r b_\phi \right) \tag{28}$$

$$j_\phi = \sigma\left(-\frac{1}{r\sin\theta}\frac{\partial \hat{\varphi}_e}{\partial \phi} - \frac{\partial a_\phi}{\partial \hat{t}} + u_r b_\theta - u_\theta b_r \right) \tag{29}$$

式(5)に示したように，回転系からみたスカラポテンシャルにはベクトルポテンシャルの φ 成分を考慮することで，Ohm の法則は座標系に依らず不変となる．本稿では，以下の誘導方程式には磁場を用いる場合を紹介しているが，この方法は，Ohm の法則の Curl を取って，ベクトルポテンシャルとスカラポテンシャルが現れないようにしたものである．一方で，Ohm の法則の Curl を取らずに，ベクトルポテンシャルとスカラポテンシャルを同時に求めていく方法も提案できる．その場合は，得られたスカラポテンシャルの値は，慣性系と回転系では異なることに注意を要する．

＜誘導方程式（B 表示）＞

$$\begin{aligned}
&\frac{\partial b_r}{\partial \hat{t}} + u_r\frac{\partial b_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial b_r}{\partial \theta} + \frac{\hat{u}_\phi}{r\sin\theta}\frac{\partial b_r}{\partial \phi} = b_r\frac{\partial u_r}{\partial r} + \frac{b_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{b_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} \\
&\quad + \nu_m\left(\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial b_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial b_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 b_r}{\partial \phi^2} \\ &-\frac{2b_r}{r^2} - \frac{2}{r^2}\frac{\partial b_\theta}{\partial \theta} - \frac{2b_\theta\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial b_\phi}{\partial \phi} \end{aligned} \right)
\end{aligned} \tag{30}$$

$$\begin{aligned}
&\frac{\partial b_\theta}{\partial \hat{t}} + u_r\frac{\partial b_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial b_\theta}{\partial \theta} + \frac{\hat{u}_\phi}{r\sin\theta}\frac{\partial b_\theta}{\partial \phi} + \frac{u_\theta b_r}{r} = b_r\frac{\partial u_\theta}{\partial r} + \frac{b_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{b_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{b_\theta u_r}{r} \\
&\quad + \nu_m\left(\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial b_\theta}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial b_\theta}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 b_\theta}{\partial \phi^2} \\ &+\frac{2}{r^2}\frac{\partial b_r}{\partial \theta} - \frac{b_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial b_\phi}{\partial \phi} \end{aligned} \right)
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\frac{\partial b_\phi}{\partial \hat{t}} + u_r\frac{\partial b_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial b_\phi}{\partial \theta} + \frac{\hat{u}_\phi}{r\sin\theta}\frac{\partial b_\phi}{\partial \phi} + \frac{\hat{u}_\phi b_r}{r} + \frac{\hat{u}_\phi b_\theta\cot\theta}{r} = b_r\frac{\partial \hat{u}_\phi}{\partial r} + \frac{b_\theta}{r}\frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{b_\phi}{r\sin\theta}\frac{\partial \hat{u}_\phi}{\partial \phi} \\
&\quad + \frac{b_\phi u_r}{r} + \frac{b_\phi u_\theta\cot\theta}{r} + \nu_m\left(\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial b_\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial b_\phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 b_\phi}{\partial \phi^2} \\ &-\frac{b_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial b_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial b_\theta}{\partial \phi} \end{aligned} \right)
\end{aligned} \tag{32}$$

＜誘導方程式（A 表示；クーロンゲージ）＞

$$
\begin{aligned}
&\frac{\partial a_r}{\partial \hat{t}}+\frac{u_\theta}{r}\frac{\partial a_r}{\partial \theta}+\frac{\hat{u}_\phi}{r\sin\theta}\frac{\partial a_r}{\partial \phi}-\frac{u_\theta a_\theta}{r}-\frac{\hat{u}_\phi a_\phi}{r}=-\frac{\partial \hat{\varphi}_e}{\partial r}+u_\theta\frac{\partial a_\theta}{\partial r}+\hat{u}_\phi\frac{\partial a_\phi}{\partial r} \\
&+\nu_m\left(\begin{array}{l}\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial a_r}{\partial r}\right)+\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial a_r}{\partial\theta}\right)+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 a_r}{\partial\phi^2} \\ -\dfrac{2a_r}{r^2}-\dfrac{2}{r^2}\dfrac{\partial a_\theta}{\partial\theta}-\dfrac{2a_\theta\cot\theta}{r^2}-\dfrac{2}{r^2\sin\theta}\dfrac{\partial a_\phi}{\partial\phi}\end{array}\right)
\end{aligned}
\tag{33}
$$

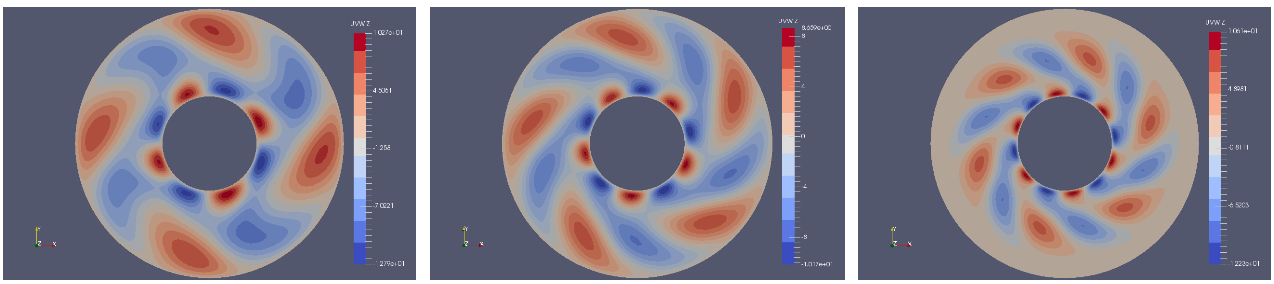
$$
\begin{aligned}
&\frac{\partial a_\theta}{\partial \hat{t}}+u_r\frac{\partial a_\theta}{\partial r}+\frac{\hat{u}_\phi}{r\sin\theta}\frac{\partial a_\theta}{\partial \phi}-\frac{\hat{u}_\phi a_\phi\cot\theta}{r}=-\frac{1}{r}\frac{\partial \hat{\varphi}_e}{\partial \theta}+u_r\left(\frac{1}{r}\frac{\partial a_r}{\partial\theta}-\frac{a_\theta}{r}\right)+\hat{u}_\phi\left(\frac{1}{r}\frac{\partial a_\phi}{\partial\theta}\right) \\
&+\nu_m\left(\begin{array}{l}\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial a_\theta}{\partial r}\right)+\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial a_\theta}{\partial\theta}\right)+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 a_\theta}{\partial\phi^2} \\ +\dfrac{2}{r^2}\dfrac{\partial a_r}{\partial\theta}-\dfrac{a_\theta}{r^2\sin^2\theta}-\dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial a_\phi}{\partial\phi}\end{array}\right)
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
&\frac{\partial a_\phi}{\partial \hat{t}}+u_r\frac{\partial a_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial a_\phi}{\partial \theta}=-\frac{1}{r\sin\theta}\frac{\partial \hat{\varphi}_e}{\partial \phi}+u_r\left(\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial\phi}-\frac{a_\phi}{r}\right)+u_\theta\left(\frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial\phi}-\frac{a_\phi\cot\theta}{r}\right) \\
&+\nu_m\left(\begin{array}{l}\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial a_\phi}{\partial r}\right)+\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial a_\phi}{\partial\theta}\right)+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 a_\phi}{\partial\phi^2} \\ -\dfrac{a_\phi}{r^2\sin^2\theta}+\dfrac{2}{r^2\sin\theta}\dfrac{\partial a_r}{\partial\phi}+\dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial a_\theta}{\partial\phi}\end{array}\right)
\end{aligned}
\tag{35}
$$

＜磁荷不在の法則あるいはクーロンゲージ＞

式(21)に同じ．

以上をまとめると，時間項のある方程式に対しても，式(4)の回転座標系における時間を導入すれば，運動方程式以外の式は不変である．運動方程式にはコリオリ力と遠心力の項が追加で現れる．実際の回転座標系における熱対流の数値計算には，式(23), (24)の重力項や遠心力項に対して，Boussinesq 近似等を用いれば良い．その後，無次元化すると，Ekman 数，Rayleigh 数，Prandtl 数，Froude 数，磁気 Prandtl 数などが現れる．図２には，半径比（外半径，内半径）が 3:1 の時の球殻熱対流の解析結果の一例を示す．



(a) Ek = 1.0×10^{-3}, Ra = 4.5×10^{4}　(b) Ek = 5.0×10^{-4}, Ra = 8.0×10^{4}　(c) Ek = 1.0×10^{-4}, Ra = 4.0×10^{5}

図２　北極側から見た赤道断面でのトロイダル方向速度成分のスナップショット（Pr = 1, Fr = 0.05）

付録　式(4)について

１．ベクトル値関数の時間変化

ベクトルの時間変化の場合は

$$\frac{\partial \vec{a}}{\partial \hat{t}} = \frac{\partial \vec{a}}{\partial t} + \left(\vec{\Omega} \times \vec{r}\right) \cdot \vec{\nabla} \vec{a} - \vec{\Omega} \times \vec{a}$$

と書ける．これが，

$$\frac{\partial \vec{a}}{\partial \hat{t}} = \frac{\partial \vec{a}}{\partial t} + \Omega \frac{\partial \vec{a}}{\partial \phi}$$

に一致するかどうか調べる．まず，

$$\vec{\Omega} \times \vec{a} = \begin{vmatrix} \vec{e}_r & \vec{e}_\theta & \vec{e}_\phi \\ \Omega \cos\theta & -\Omega \sin\theta & 0 \\ a_r & a_\theta & a_\phi \end{vmatrix} = \left(-a_\phi \Omega \sin\theta\right)\vec{e}_r + \left(-a_\phi \Omega \cos\theta\right)\vec{e}_\theta + \left(a_\theta \Omega \cos\theta + a_r \Omega \sin\theta\right)\vec{e}_\phi$$

と求まる．次に，$\left(\vec{\Omega} \times \vec{r}\right) \cdot \vec{\nabla} \vec{a}$ に対しては，以下の公式を利用する．

$$\begin{aligned} \vec{b} \cdot \vec{\nabla} \vec{u} = & \left(b_r \frac{\partial u_r}{\partial r} + \frac{b_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{b_\theta u_\theta}{r} + \frac{b_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{b_\phi u_\phi}{r}\right)\vec{e}_r \\ & + \left(b_r \frac{\partial u_\theta}{\partial r} + \frac{b_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{b_\theta u_r}{r} + \frac{b_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{b_\phi u_\phi \cot\theta}{r}\right)\vec{e}_\theta \\ & + \left(b_r \frac{\partial u_\phi}{\partial r} + \frac{b_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{b_\phi u_r}{r} + \frac{b_\phi u_\theta \cos\theta}{r\sin\theta} + \frac{b_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right)\vec{e}_\phi \end{aligned}$$

ここで，$\vec{b} = \vec{\Omega} \times \vec{r}$，$\vec{u} = \vec{a}$ と置く．ただし，$\vec{\Omega} \times \vec{r} = \left(r\Omega \sin\theta\right)\vec{e}_\phi$　に留意する．

$$\begin{aligned} \left(\vec{\Omega} \times \vec{r}\right) \cdot \vec{\nabla} \vec{a} = & \left(\frac{r\Omega \sin\theta}{r\sin\theta}\frac{\partial a_r}{\partial \phi} - \frac{r\Omega \sin\theta a_\phi}{r}\right)\vec{e}_r + \left(\frac{r\Omega \sin\theta}{r\sin\theta}\frac{\partial a_\theta}{\partial \phi} - \frac{r\Omega \sin\theta a_\phi \cot\theta}{r}\right)\vec{e}_\theta \\ & + \left(\frac{r\Omega \sin\theta a_r}{r} + \frac{r\Omega \sin\theta a_\theta \cos\theta}{r\sin\theta} + \frac{r\Omega \sin\theta}{r\sin\theta}\frac{\partial a_\phi}{\partial \phi}\right)\vec{e}_\phi \end{aligned}$$

$$\left(\vec{\Omega} \times \vec{r}\right) \cdot \vec{\nabla} \vec{a} = \Omega\left(\frac{\partial a_r}{\partial \phi} - a_\phi \sin\theta\right)\vec{e}_r + \Omega\left(\frac{\partial a_\theta}{\partial \phi} - a_\phi \cos\theta\right)\vec{e}_\theta + \Omega\left(a_r \sin\theta + a_\theta \cos\theta + \frac{\partial a_\phi}{\partial \phi}\right)\vec{e}_\phi$$

２つの項を同時に考慮すると，

$$\left(\vec{\Omega}\times\vec{r}\right)\cdot\vec{\nabla}\vec{a}-\vec{\Omega}\times\vec{a}$$

$$=\Omega\left(\frac{\partial a_r}{\partial\phi}-a_\phi\sin\theta\right)\vec{e}_r+\Omega\left(\frac{\partial a_\theta}{\partial\phi}-a_\phi\cos\theta\right)\vec{e}_\theta+\Omega\left(a_r\sin\theta+a_\theta\cos\theta+\frac{\partial a_\phi}{\partial\phi}\right)\vec{e}_\phi$$

$$-\Omega\left(-a_\phi\sin\theta\right)\vec{e}_r-\Omega\left(-a_\phi\cos\theta\right)\vec{e}_\theta-\Omega\left(a_\theta\cos\theta+a_r\sin\theta\right)\vec{e}_\phi$$

$$=\Omega\left(\frac{\partial a_r}{\partial\phi}\vec{e}_r+\frac{\partial a_\theta}{\partial\phi}\vec{e}_\theta+\frac{\partial a_\phi}{\partial\phi}\vec{e}_\phi\right)=\Omega\frac{\partial\vec{a}}{\partial\phi}$$

したがって，

$$\frac{\partial\vec{a}}{\partial\hat{t}}=\frac{\partial\vec{a}}{\partial t}+\left(\vec{\Omega}\times\vec{r}\right)\cdot\vec{\nabla}\vec{a}-\vec{\Omega}\times\vec{a}=\frac{\partial\vec{a}}{\partial t}+\Omega\frac{\partial\vec{a}}{\partial\phi}$$

となることがわかる．

２．スカラ値関数の時間変化

スカラの時間変化では$\vec{\Omega}\times$の項は省かれ

$$\frac{\partial\varphi}{\partial\hat{t}}=\frac{\partial\varphi}{\partial t}+\left(\vec{\Omega}\times\vec{r}\right)\cdot\vec{\nabla}\varphi$$

と書ける．これが，

$$\frac{\partial\varphi}{\partial\hat{t}}=\frac{\partial\varphi}{\partial t}+\Omega\frac{\partial\varphi}{\partial\phi}$$

に一致するかどうか調べる．

$$\left(\vec{\Omega}\times\vec{r}\right)\cdot\vec{\nabla}=\left(\vec{e}_\phi\Omega r\sin\theta\right)\cdot\left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)=\Omega\frac{\partial}{\partial\phi}$$

したがって，

$$\frac{\partial\varphi}{\partial\hat{t}}=\frac{\partial\varphi}{\partial t}+\left(\vec{\Omega}\times\vec{r}\right)\cdot\vec{\nabla}\varphi=\frac{\partial\varphi}{\partial t}+\Omega\frac{\partial\varphi}{\partial\phi}$$

となる．

結局，この問題では，ベクトル，スカラのどちらの関数が作用する場合でも，回転系での時間変化は

$$\frac{\partial}{\partial\hat{t}}=\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}$$

と書けることになる．