円筒座標系におけるテンソル

＜デカルト座標系$\left(x,y,z\right)$と円筒座標系$\left(r,\theta,z\right)$＞

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} 0 \le r \le +\infty \\ 0 \le \theta \le 2\pi \\ -\infty \le z \le +\infty \end{cases} \tag{1}$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z \tag{2}$$

＜単位ベクトル＞

$$\begin{cases} \vec{e}_r = \left(\cos\theta\right)\vec{e}_x + \left(\sin\theta\right)\vec{e}_y + \left(0\right)\vec{e}_z \\ \vec{e}_\theta = \left(-\sin\theta\right)\vec{e}_x + \left(\cos\theta\right)\vec{e}_y + \left(0\right)\vec{e}_z \\ \vec{e}_z = \left(0\right)\vec{e}_x + \left(0\right)\vec{e}_y + \left(1\right)\vec{e}_z \end{cases} \tag{3a}$$

$$\begin{cases} \vec{e}_x = \left(\cos\theta\right)\vec{e}_r + \left(-\sin\theta\right)\vec{e}_\theta + \left(0\right)\vec{e}_z \\ \vec{e}_y = \left(\sin\theta\right)\vec{e}_r + \left(\cos\theta\right)\vec{e}_\theta + \left(0\right)\vec{e}_z \\ \vec{e}_z = \left(0\right)\vec{e}_r + \left(0\right)\vec{e}_\theta + \left(1\right)\vec{e}_z \end{cases} \tag{3b}$$

＜単位ベクトルの偏導関数＞

$$\begin{cases} \dfrac{\partial \vec{e}_r}{\partial r} = \vec{0}, & \dfrac{\partial \vec{e}_\theta}{\partial r} = \vec{0}, & \dfrac{\partial \vec{e}_z}{\partial r} = \vec{0} \\ \dfrac{\partial \vec{e}_r}{\partial \theta} = \vec{e}_\theta, & \dfrac{\partial \vec{e}_\theta}{\partial \theta} = -\vec{e}_r, & \dfrac{\partial \vec{e}_z}{\partial \theta} = \vec{0} \\ \dfrac{\partial \vec{e}_r}{\partial z} = \vec{0}, & \dfrac{\partial \vec{e}_\theta}{\partial z} = \vec{0}, & \dfrac{\partial \vec{e}_z}{\partial z} = \vec{0} \end{cases} \tag{4}$$

＜位置ベクトル＞

$$\vec{r} = r\vec{e}_r + z\vec{e}_z$$

$$\Rightarrow \quad d\vec{r} = \left(dr\right)\vec{e}_r + r\underbrace{\left(d\vec{e}_r\right)}_{\frac{\partial \vec{e}_r}{\partial \theta}d\theta} + \left(dz\right)\vec{e}_z + z\underbrace{\left(d\vec{e}_z\right)}_{0} = \left(dr\right)\vec{e}_r + \left(rd\theta\right)\vec{e}_\theta + \left(dz\right)\vec{e}_z \tag{5}$$

以下，式(4)に注意しながら演算を進める．

＜勾配＞

$$\vec{\nabla} = \vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z} = \vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial \theta} + \vec{e}_z\frac{\partial}{\partial z} \tag{6}$$

＜ラプラシアン＞

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\vec{e}_z\frac{\partial}{\partial z}\right)\cdot\left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\vec{e}_z\frac{\partial}{\partial z}\right)$$
$$=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2} \tag{7}$$

＜ベクトル場の発散＞

$$\vec{\nabla}\cdot\vec{u} = \left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\vec{e}_z\frac{\partial}{\partial z}\right)\cdot\left(\vec{e}_r u_r+\vec{e}_\theta u_\theta+\vec{e}_z u_z\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z} \tag{8}$$

＜実質微分＞

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}=\frac{\partial}{\partial t}+\left(\vec{e}_r u_r+\vec{e}_\theta u_\theta+\vec{e}_z u_z\right)\cdot\left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\vec{e}_z\frac{\partial}{\partial z}\right)$$
$$=\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z} \tag{9}$$

＜ベクトル場の実質微分＞

式(9)の結果を用いる．

$$\frac{D\vec{b}}{Dt}=\frac{\partial\vec{b}}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\vec{b}=\left(\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}\right)\left(\vec{e}_r b_r+\vec{e}_\theta b_\theta+\vec{e}_z b_z\right)$$
$$=\vec{e}_r\left(\frac{\partial b_r}{\partial t}+u_r\frac{\partial b_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_r}{\partial\theta}+u_z\frac{\partial b_r}{\partial z}-\frac{u_\theta b_\theta}{r}\right)$$
$$+\vec{e}_\theta\left(\frac{\partial b_\theta}{\partial t}+u_r\frac{\partial b_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_\theta}{\partial\theta}+u_z\frac{\partial b_\theta}{\partial z}+\frac{u_\theta b_r}{r}\right) \tag{10}$$
$$+\vec{e}_z\left(\frac{\partial b_z}{\partial t}+u_r\frac{\partial b_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_z}{\partial\theta}+u_z\frac{\partial b_z}{\partial z}\right)$$

＜ベクトル場の回転＞

$$\vec{\nabla}\times\vec{u}=\left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\vec{e}_z\frac{\partial}{\partial z}\right)\times\left(\vec{e}_r u_r+\vec{e}_\theta u_\theta+\vec{e}_z u_z\right)$$
$$=\frac{\vec{e}_r}{r}\left\{\frac{\partial u_z}{\partial\theta}-\frac{\partial\left(ru_\theta\right)}{\partial z}\right\}+\vec{e}_\theta\left\{\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right\}+\frac{\vec{e}_z}{r}\left\{\frac{\partial\left(ru_\theta\right)}{\partial r}-\frac{\partial u_r}{\partial\theta}\right\}=\begin{vmatrix}\frac{\vec{e}_r}{r} & \vec{e}_\theta & \frac{\vec{e}_z}{r}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ u_r & ru_\theta & u_z\end{vmatrix} \tag{11}$$

＜ベクトル場の勾配＞

$$\vec{\nabla}\vec{u}=\left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\vec{e}_z\frac{\partial}{\partial z}\right)\left(\vec{e}_r u_r+\vec{e}_\theta u_\theta+\vec{e}_z u_z\right)$$
$$=\vec{e}_r\vec{e}_r\frac{\partial u_r}{\partial r}+\vec{e}_r\vec{e}_\theta\frac{\partial u_\theta}{\partial r}+\vec{e}_r\vec{e}_z\frac{\partial u_z}{\partial r}\quad+\vec{e}_\theta\vec{e}_r\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)+\vec{e}_\theta\vec{e}_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+\vec{e}_\theta\vec{e}_z\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right) \tag{12}$$
$$+\vec{e}_z\vec{e}_r\left(\frac{\partial u_r}{\partial z}\right)+\vec{e}_z\vec{e}_\theta\left(\frac{\partial u_\theta}{\partial z}\right)+\vec{e}_z\vec{e}_z\left(\frac{\partial u_z}{\partial z}\right)$$

＜ベクトル場のラプラシアン＞

式(12)の発散をとる．

$$\nabla^2\vec{u}=\vec{\nabla}\cdot\left(\vec{\nabla}\vec{u}\right)=\left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\vec{e}_z\frac{\partial}{\partial z}\right)$$

$$\cdot\left(\begin{array}{l}\vec{e}_r\vec{e}_r\dfrac{\partial u_r}{\partial r}+\vec{e}_r\vec{e}_\theta\dfrac{\partial u_\theta}{\partial r}+\vec{e}_r\vec{e}_z\dfrac{\partial u_z}{\partial r}+\vec{e}_\theta\vec{e}_r\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}-\dfrac{u_\theta}{r}\right)+\vec{e}_\theta\vec{e}_\theta\left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta}+\dfrac{u_r}{r}\right)+\vec{e}_\theta\vec{e}_z\left(\dfrac{1}{r}\dfrac{\partial u_z}{\partial\theta}\right)\\ \quad+\vec{e}_z\vec{e}_r\left(\dfrac{\partial u_r}{\partial z}\right)+\vec{e}_z\vec{e}_\theta\left(\dfrac{\partial u_\theta}{\partial z}\right)+\vec{e}_z\vec{e}_z\left(\dfrac{\partial u_z}{\partial z}\right)\end{array}\right)$$

$$\begin{aligned}&=\vec{e}_r\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2u_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2u_r}{\partial z^2}\right)\\&\quad+\vec{e}_\theta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{\partial^2u_\theta}{\partial z^2}\right)\\&\quad+\vec{e}_z\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2u_z}{\partial\theta^2}+\frac{\partial^2u_z}{\partial z^2}\right)\end{aligned}\tag{13}$$

＜ベクトル場の実質微分（その２）＞

式(12)の結果を使う．

$$\vec{b}\cdot\vec{\nabla}\vec{u}=\left(\vec{e}_rb_r+\vec{e}_\theta b_\theta+\vec{e}_zb_z\right)$$

$$\cdot\left(\begin{array}{l}\vec{e}_r\vec{e}_r\dfrac{\partial u_r}{\partial r}+\vec{e}_r\vec{e}_\theta\dfrac{\partial u_\theta}{\partial r}+\vec{e}_r\vec{e}_z\dfrac{\partial u_z}{\partial r}+\vec{e}_\theta\vec{e}_r\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}-\dfrac{u_\theta}{r}\right)+\vec{e}_\theta\vec{e}_\theta\left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta}+\dfrac{u_r}{r}\right)+\vec{e}_\theta\vec{e}_z\left(\dfrac{1}{r}\dfrac{\partial u_z}{\partial\theta}\right)\\ \quad+\vec{e}_z\vec{e}_r\left(\dfrac{\partial u_r}{\partial z}\right)+\vec{e}_z\vec{e}_\theta\left(\dfrac{\partial u_\theta}{\partial z}\right)+\vec{e}_z\vec{e}_z\left(\dfrac{\partial u_z}{\partial z}\right)\end{array}\right)$$

$$\begin{aligned}&=\vec{e}_rb_r\frac{\partial u_r}{\partial r}+\vec{e}_\theta b_r\frac{\partial u_\theta}{\partial r}+\vec{e}_zb_r\frac{\partial u_z}{\partial r}+\vec{e}_rb_\theta\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)+\vec{e}_\theta b_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+\vec{e}_zb_\theta\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)\\&\quad+\vec{e}_rb_z\left(\frac{\partial u_r}{\partial z}\right)+\vec{e}_\theta b_z\left(\frac{\partial u_\theta}{\partial z}\right)+\vec{e}_zb_z\left(\frac{\partial u_z}{\partial z}\right)\end{aligned}$$

$$\begin{aligned}&=\vec{e}_r\left\{b_r\frac{\partial u_r}{\partial r}+b_\theta\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)+b_z\left(\frac{\partial u_r}{\partial z}\right)\right\}+\vec{e}_\theta\left\{b_r\frac{\partial u_\theta}{\partial r}+b_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+b_z\left(\frac{\partial u_\theta}{\partial z}\right)\right\}\\&\quad+\vec{e}_z\left\{b_r\frac{\partial u_z}{\partial r}+b_\theta\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)+b_z\left(\frac{\partial u_z}{\partial z}\right)\right\}\end{aligned}\tag{14}$$

これは，式(10)に示される結果と同じ．

＜粘性応力テンソル＞

式(12)の結果から

$$\frac{\boldsymbol{\tau}}{\mu}=\vec{e}_r\vec{e}_r\left\{2\left(-\frac{\Theta}{3}+\frac{\partial u_r}{\partial r}\right)\right\}+\vec{e}_r\vec{e}_\theta\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right)+\vec{e}_r\vec{e}_z\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)$$
$$+\vec{e}_\theta\vec{e}_r\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right)+\vec{e}_\theta\vec{e}_\theta\left\{2\left(-\frac{\Theta}{3}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)\right\}+\vec{e}_\theta\vec{e}_z\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right) \qquad (15)$$
$$+\vec{e}_z\vec{e}_r\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)+\vec{e}_z\vec{e}_\theta\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)+\vec{e}_z\vec{e}_z\left\{2\left(-\frac{\Theta}{3}+\frac{\partial u_z}{\partial z}\right)\right\}$$

ただし，$\Theta=\vec{\nabla}\cdot\vec{u}=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}$

以下，電磁流体力学に必要となる基礎方程式（成分表示）を示す．

＜質量保存式＞

$$\frac{\partial \rho}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\rho=-\rho\vec{\nabla}\cdot\vec{u}$$

式(8), (9)を使って，

$$\frac{\partial \rho}{\partial t}+u_r\frac{\partial \rho}{\partial r}+\frac{u_\theta}{r}\frac{\partial \rho}{\partial \theta}+u_z\frac{\partial \rho}{\partial z}=-\rho\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right\} \qquad (16)$$

＜運動方程式＞

$$\rho\left\{\frac{\partial \vec{u}}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\right\}=-\vec{\nabla}p+\mu\nabla^2\vec{u}+\vec{j}\times\vec{b}+\rho\vec{g}$$

式(10), (13)を使って，

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}-\frac{{u_\theta}^2}{r}\right)=-\frac{\partial p}{\partial r}+\left(j_\theta b_z-j_z b_\theta\right)+\rho g_r$$
$$+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right) \qquad (17)$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_\theta u_r}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\left(j_z b_r-j_r b_z\right)+\rho g_\theta$$
$$+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right) \qquad (18)$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\left(j_r b_\theta-j_\theta b_r\right)+\rho g_z$$
$$+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right) \qquad (19)$$

＜エネルギー方程式＞

$$\frac{DT}{Dt}=\alpha\nabla^2T+\Phi$$

式(7), (9), (15)を使って，

$$\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial \theta}+u_z\frac{\partial T}{\partial z}=\alpha\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right\}$$
$$+\mu\left[\begin{array}{l}2\left(\frac{\partial u_r}{\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)^2+2\left(\frac{\partial u_z}{\partial z}\right)^2\\ +\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right)^2+\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right)^2\\ -\frac{2}{3}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right\}^2\end{array}\right] \tag{20}$$

＜電荷保存則＞

$$\vec{\nabla}\cdot\vec{j}=0$$

式(8)から

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rj_r\right)+\frac{1}{r}\frac{\partial j_\theta}{\partial \theta}+\frac{\partial j_z}{\partial z}=0 \tag{21}$$

＜Ohm の法則＞

$$\vec{j}=\sigma\left(-\vec{\nabla}\varphi-\frac{\partial \vec{a}}{\partial t}+\vec{u}\times\vec{b}\right)$$

式(6)から

$$\vec{j}=\underbrace{\sigma\left(-\frac{\partial \varphi}{\partial r}-\frac{\partial a_r}{\partial t}+u_\theta b_z-u_z b_\theta\right)}_{j_r}\vec{e}_r+\underbrace{\sigma\left(-\frac{1}{r}\frac{\partial \varphi}{\partial \theta}-\frac{\partial a_\theta}{\partial t}+u_z b_r-u_r b_z\right)}_{j_\theta}\vec{e}_\theta$$
$$+\underbrace{\sigma\left(-\frac{\partial \varphi}{\partial z}-\frac{\partial a_z}{\partial t}+u_r b_\theta-u_\theta b_r\right)}_{j_z}\vec{e}_z \tag{22}$$

＜誘導方程式 B 表示＞

$$\frac{\partial \vec{b}}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\vec{b}=\left(\vec{b}\cdot\vec{\nabla}\right)\vec{u}+\nu_m\nabla^2\vec{b}$$

式(10), (13), (14)を使って

$$\frac{\partial b_r}{\partial t}+u_r\frac{\partial b_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_r}{\partial \theta}+u_z\frac{\partial b_r}{\partial z}=b_r\frac{\partial u_r}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_r}{\partial \theta}+b_z\frac{\partial u_r}{\partial z}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_r}{\partial r}\right)-\frac{b_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 b_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial b_\theta}{\partial \theta}+\frac{\partial^2 b_r}{\partial z^2}\right) \tag{23}$$

$$\frac{\partial b_\theta}{\partial t}+u_r\frac{\partial b_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_\theta}{\partial \theta}+u_z\frac{\partial b_\theta}{\partial z}+\frac{u_\theta b_r}{r}=b_r\frac{\partial u_\theta}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+b_z\frac{\partial u_\theta}{\partial z}+\frac{b_\theta u_r}{r}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_\theta}{\partial r}\right)-\frac{b_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 b_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial b_r}{\partial \theta}+\frac{\partial^2 b_\theta}{\partial z^2}\right) \tag{24}$$

$$\frac{\partial b_z}{\partial t}+u_r\frac{\partial b_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_z}{\partial \theta}+u_z\frac{\partial b_z}{\partial z}=b_r\frac{\partial u_z}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_z}{\partial \theta}+b_z\frac{\partial u_z}{\partial z}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 b_z}{\partial \theta^2}+\frac{\partial^2 b_z}{\partial z^2}\right) \tag{25}$$

＜誘導方程式 A-φ 表示＞

$$\frac{\partial \vec{a}}{\partial t}=-\vec{\nabla}\varphi+\vec{u}\times\left(\vec{\nabla}\times\vec{a}\right)+\nu_m\nabla^2\vec{a}$$

ここで，右辺第 2 項は

$$\vec{u}\times\left(\vec{\nabla}\times\vec{a}\right)=\varepsilon_{ijk}u_j\left(\varepsilon_{klm}\partial_l a_m\right)=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)u_j\partial_l a_m=u_j\partial_i a_j-u_j\partial_j a_i$$

であるから，誘導方程式の A 表示は次式と等価である．

$$\frac{\partial a_i}{\partial t}+u_j\frac{\partial a_i}{\partial x_j}=-\frac{\partial \varphi}{\partial x_i}+u_j\frac{\partial a_j}{\partial x_i}+\nu_m\nabla^2 a_i$$

まとめると，3 成分は以下のように書ける．

$$\frac{\partial a_r}{\partial t}+\frac{u_\theta}{r}\frac{\partial a_r}{\partial \theta}+u_z\frac{\partial a_r}{\partial z}=-\frac{\partial \varphi}{\partial r}+u_\theta\frac{\partial a_\theta}{\partial r}+u_z\frac{\partial a_z}{\partial r}+\frac{u_\theta a_\theta}{r}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a_r}{\partial r}\right)-\frac{a_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 a_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial a_\theta}{\partial \theta}+\frac{\partial^2 a_r}{\partial z^2}\right) \tag{26}$$

$$\frac{\partial a_\theta}{\partial t}+u_r\frac{\partial a_\theta}{\partial r}+u_z\frac{\partial a_\theta}{\partial z}=-\frac{1}{r}\frac{\partial \varphi}{\partial \theta}+\frac{u_r}{r}\frac{\partial a_r}{\partial \theta}+\frac{u_z}{r}\frac{\partial a_z}{\partial \theta}-\frac{u_r a_\theta}{r}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a_\theta}{\partial r}\right)-\frac{a_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 a_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial a_r}{\partial \theta}+\frac{\partial^2 a_\theta}{\partial z^2}\right) \tag{27}$$

$$\frac{\partial a_z}{\partial t}+u_r\frac{\partial a_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial a_z}{\partial \theta}=-\frac{\partial \varphi}{\partial z}+u_r\frac{\partial a_r}{\partial z}+u_\theta\frac{\partial a_\theta}{\partial z}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 a_z}{\partial \theta^2}+\frac{\partial^2 a_z}{\partial z^2}\right) \tag{28}$$

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