球座標系におけるテンソル

＜デカルト座標系(x,y,z)と球座標系(r,θ,ϕ)＞

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} 0 \le r \le +\infty \\ 0 \le \theta \le \pi \\ 0 \le \phi \le 2\pi \end{cases} \tag{1}$$

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right), \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{2}$$

＜単位ベクトル＞

$$\begin{cases} \vec{e}_r = (\sin\theta\cos\phi)\vec{e}_x + (\sin\theta\sin\phi)\vec{e}_y + (\cos\theta)\vec{e}_z \\ \vec{e}_\theta = (\cos\theta\cos\phi)\vec{e}_x + (\cos\theta\sin\phi)\vec{e}_y + (-\sin\theta)\vec{e}_z \\ \vec{e}_\phi = (-\sin\phi)\vec{e}_x + (\cos\phi)\vec{e}_y + (0)\vec{e}_z \end{cases} \tag{3a}$$

$$\begin{cases} \vec{e}_x = (\sin\theta\cos\phi)\vec{e}_r + (\cos\theta\cos\phi)\vec{e}_\theta + (-\sin\phi)\vec{e}_\phi \\ \vec{e}_y = (\sin\theta\sin\phi)\vec{e}_r + (\cos\theta\sin\phi)\vec{e}_\theta + (\cos\phi)\vec{e}_\phi \\ \vec{e}_z = (\cos\theta)\vec{e}_r + (-\sin\theta)\vec{e}_\theta + (0)\vec{e}_\phi \end{cases} \tag{3b}$$

＜単位ベクトルの偏導関数＞

$$\begin{cases} \dfrac{\partial\vec{e}_r}{\partial r} = \vec{0}, & \dfrac{\partial\vec{e}_\theta}{\partial r} = \vec{0}, & \dfrac{\partial\vec{e}_\phi}{\partial r} = \vec{0} \\ \dfrac{\partial\vec{e}_r}{\partial\theta} = \vec{e}_\theta, & \dfrac{\partial\vec{e}_\theta}{\partial\theta} = -\vec{e}_r, & \dfrac{\partial\vec{e}_\phi}{\partial\theta} = \vec{0} \\ \dfrac{\partial\vec{e}_r}{\partial\phi} = (\sin\theta)\vec{e}_\phi, & \dfrac{\partial\vec{e}_\theta}{\partial\phi} = (\cos\theta)\vec{e}_\phi, & \dfrac{\partial\vec{e}_\phi}{\partial\phi} = (-\sin\theta)\vec{e}_r + (-\cos\theta)\vec{e}_\theta \end{cases} \tag{4}$$

＜位置ベクトル＞

$$\begin{aligned} \vec{r} = r\vec{e}_r \quad \Rightarrow \quad d\vec{r} &= (dr)\vec{e}_r + r(d\vec{e}_r) = (dr)\vec{e}_r + r\left(\frac{\partial\vec{e}_r}{\partial r}dr + \underbrace{\frac{\partial\vec{e}_r}{\partial\theta}}_{\vec{e}_\theta}d\theta + \underbrace{\frac{\partial\vec{e}_r}{\partial\phi}}_{(\sin\theta)\vec{e}_\phi}d\phi\right) \\ &= (dr)\vec{e}_r + (rd\theta)\vec{e}_\theta + (r\sin\theta d\phi)\vec{e}_\phi \end{aligned} \tag{5}$$

以下，式(4)に注意しながら演算を進める．

＜勾配＞

$$\vec{\nabla} = \vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z} = \vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{6}$$

＜ラプラシアン＞

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \left(\vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{7}$$

＜ベクトル場の発散＞

$$\vec{\nabla}\cdot\vec{u} = \left(\vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\vec{e}_r u_r + \vec{e}_\theta u_\theta + \vec{e}_\phi u_\phi\right)$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial\left(u_\theta\sin\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} \tag{8}$$

＜実質微分＞

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla} = \frac{\partial}{\partial t} + \left(\vec{e}_r u_r + \vec{e}_\theta u_\theta + \vec{e}_\phi u_\phi\right)\cdot\left(\vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$
$$= \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{9}$$

＜ベクトル場の実質微分＞

式(9)の結果を用いる．

$$\frac{D\vec{b}}{Dt} = \frac{\partial\vec{b}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{b} = \left(\frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\vec{e}_r b_r + \vec{e}_\theta b_\theta + \vec{e}_\phi b_\phi\right)$$
$$= \vec{e}_r\left(\frac{\partial b_r}{\partial t} + u_r\frac{\partial b_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial b_r}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial b_r}{\partial\phi} - \frac{u_\theta b_\theta + u_\phi b_\phi}{r}\right)$$
$$+ \vec{e}_\theta\left(\frac{\partial b_\theta}{\partial t} + u_r\frac{\partial b_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial b_\theta}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial b_\theta}{\partial\phi} + \frac{u_\theta b_r}{r} - \frac{u_\phi b_\phi}{r}\cot\theta\right) \tag{10}$$
$$+ \vec{e}_\phi\left(\frac{\partial b_\phi}{\partial t} + u_r\frac{\partial b_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial b_\phi}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial b_\phi}{\partial\phi} + \frac{u_\phi b_r}{r} + \frac{u_\phi b_\theta}{r}\cot\theta\right)$$

＜ベクトル場の回転＞

$$
\begin{aligned}
\vec{\nabla}\times\vec{u} &= \left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\times\left(\vec{e}_r u_r+\vec{e}_\theta u_\theta+\vec{e}_\phi u_\phi\right)\\
&=\vec{e}_r\left\{\frac{1}{r\sin\theta}\frac{\partial\left(u_\phi\sin\theta\right)}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial\left(u_\theta\right)}{\partial\phi}\right\}+\vec{e}_\theta\left\{\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{1}{r}\frac{\partial\left(ru_\phi\right)}{\partial r}\right\}+\vec{e}_\phi\left\{\frac{1}{r}\frac{\partial\left(ru_\theta\right)}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right\}\\
&=\frac{\vec{e}_r}{r^2\sin\theta}\left\{\frac{\partial\left(ru_\phi\sin\theta\right)}{\partial\theta}-\frac{\partial\left(ru_\theta\right)}{\partial\phi}\right\}+\frac{\vec{e}_\theta}{r\sin\theta}\left\{\frac{\partial u_r}{\partial\phi}-\frac{\partial\left(ru_\phi\sin\theta\right)}{\partial r}\right\}+\frac{\vec{e}_\phi}{r}\left\{\frac{\partial\left(ru_\theta\right)}{\partial r}-\frac{\partial u_r}{\partial\theta}\right\} \qquad (11)\\
&=\begin{vmatrix}\dfrac{\vec{e}_r}{r^2\sin\theta} & \dfrac{\vec{e}_\theta}{r\sin\theta} & \dfrac{\vec{e}_\phi}{r}\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi}\\ u_r & ru_\theta & ru_\phi\sin\theta\end{vmatrix}
\end{aligned}
$$

＜ベクトル場の勾配＞

$$
\begin{aligned}
\vec{\nabla}\vec{u} &= \left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\vec{e}_r u_r+\vec{e}_\theta u_\theta+\vec{e}_\phi u_\phi\right)\\
&=\vec{e}_r\vec{e}_r\frac{\partial u_r}{\partial r}+\vec{e}_r\vec{e}_\theta\frac{\partial u_\theta}{\partial r}+\vec{e}_r\vec{e}_\phi\frac{\partial u_\phi}{\partial r}\\
&\quad+\vec{e}_\theta\vec{e}_r\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)+\vec{e}_\theta\vec{e}_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+\vec{e}_\theta\vec{e}_\phi\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}\right) \qquad (12)\\
&\quad+\vec{e}_\phi\vec{e}_r\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}\right)+\vec{e}_\phi\vec{e}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{u_\phi\cot\theta}{r}\right)+\vec{e}_\phi\vec{e}_\phi\left(\frac{u_r}{r}+\frac{u_\theta\cos\theta}{r\sin\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)
\end{aligned}
$$

＜ベクトル場のラプラシアン＞

式(12)の発散をとる．

$$
\begin{aligned}
\nabla^2\vec{u} &= \vec{\nabla}\cdot\left(\vec{\nabla}\vec{u}\right)=\left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\\
&\cdot\left(\begin{array}{l}\vec{e}_r\vec{e}_r\dfrac{\partial u_r}{\partial r}+\vec{e}_r\vec{e}_\theta\dfrac{\partial u_\theta}{\partial r}+\vec{e}_r\vec{e}_\phi\dfrac{\partial u_\phi}{\partial r}+\vec{e}_\theta\vec{e}_r\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}-\dfrac{u_\theta}{r}\right)+\vec{e}_\theta\vec{e}_\theta\left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta}+\dfrac{u_r}{r}\right)+\vec{e}_\theta\vec{e}_\phi\left(\dfrac{1}{r}\dfrac{\partial u_\phi}{\partial\theta}\right)\\
+\vec{e}_\phi\vec{e}_r\left(\dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial\phi}-\dfrac{u_\phi}{r}\right)+\vec{e}_\phi\vec{e}_\theta\left(\dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial\phi}-\dfrac{u_\phi\cot\theta}{r}\right)+\vec{e}_\phi\vec{e}_\phi\left(\dfrac{u_r}{r}+\dfrac{u_\theta\cos\theta}{r\sin\theta}+\dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial\phi}\right)\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
&=\vec{e}_r\left(\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2}}_{\nabla^2 u_r}-\frac{2u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2u_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)\\
&+\vec{e}_\theta\left(\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\theta}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\phi^2}}_{\nabla^2 u_\theta}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right)\\
&+\vec{e}_\phi\left(\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}}_{\nabla^2 u_\phi}-\frac{u_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\right)
\end{aligned}
\tag{13}
$$

＜ベクトル場の実質微分（その２）＞

式(12)の結果を使う.

$$
\begin{aligned}
\vec{b}\cdot\vec{\nabla}\vec{u}&=\left(\vec{e}_r b_r+\vec{e}_\theta b_\theta+\vec{e}_\phi b_\phi\right)\\
&\cdot\left(\begin{aligned}&\vec{e}_r\vec{e}_r\frac{\partial u_r}{\partial r}+\vec{e}_r\vec{e}_\theta\frac{\partial u_\theta}{\partial r}+\vec{e}_r\vec{e}_\phi\frac{\partial u_\phi}{\partial r}+\vec{e}_\theta\vec{e}_r\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)+\vec{e}_\theta\vec{e}_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+\vec{e}_\theta\vec{e}_\phi\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}\right)\\
&+\vec{e}_\phi\vec{e}_r\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}\right)+\vec{e}_\phi\vec{e}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{u_\phi\cot\theta}{r}\right)+\vec{e}_\phi\vec{e}_\phi\left(\frac{u_r}{r}+\frac{u_\theta\cos\theta}{r\sin\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)\end{aligned}\right)
\end{aligned}
$$

$$
\begin{aligned}
&=\vec{e}_r b_r\frac{\partial u_r}{\partial r}+\vec{e}_\theta b_r\frac{\partial u_\theta}{\partial r}+\vec{e}_\phi b_r\frac{\partial u_\phi}{\partial r}+\vec{e}_r b_\theta\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)+\vec{e}_\theta b_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+\vec{e}_\phi b_\theta\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}\right)\\
&+\vec{e}_r b_\phi\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}\right)+\vec{e}_\theta b_\phi\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{u_\phi\cot\theta}{r}\right)+\vec{e}_\phi b_\phi\left(\frac{u_r}{r}+\frac{u_\theta\cos\theta}{r\sin\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)
\end{aligned}
$$

$$
\begin{aligned}
&=\vec{e}_r\left\{b_r\frac{\partial u_r}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{b_\theta u_\theta}{r}+\frac{b_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{b_\phi u_\phi}{r}\right\}\\
&+\vec{e}_\theta\left\{b_r\frac{\partial u_\theta}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{b_\theta u_r}{r}+\frac{b_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{b_\phi u_\phi\cot\theta}{r}\right\}\\
&+\vec{e}_\phi\left\{b_r\frac{\partial u_\phi}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_\phi}{\partial\theta}+\frac{b_\phi u_r}{r}+\frac{b_\phi u_\theta\cos\theta}{r\sin\theta}+\frac{b_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right\}
\end{aligned}
\tag{14}
$$

これは，式(10)に示される結果と同じ.

＜粘性応力テンソル＞

式(12)の結果から

$$
\begin{aligned}
\frac{\boldsymbol{\tau}}{\mu} &= \vec{e}_r\vec{e}_r\left\{2\left(-\frac{\Theta}{3}+\frac{\partial u_r}{\partial r}\right)\right\}+\vec{e}_r\vec{e}_\theta\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right)+\vec{e}_r\vec{e}_\phi\left(\frac{\partial u_\phi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r}\right)\\
&+\vec{e}_\theta\vec{e}_r\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right)+\vec{e}_\theta\vec{e}_\theta\left\{2\left(-\frac{\Theta}{3}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)\right\}+\vec{e}_\theta\vec{e}_\phi\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}-\frac{u_\phi\cot\theta}{r}\right)\\
&+\vec{e}_\phi\vec{e}_r\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r}+\frac{\partial u_\phi}{\partial r}\right)+\vec{e}_\phi\vec{e}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}-\frac{u_\phi\cot\theta}{r}+\frac{1}{r}\frac{\partial u_\phi}{\partial \theta}\right)\\
&+\vec{e}_\phi\vec{e}_\phi\left\{2\left(-\frac{\Theta}{3}+\frac{u_r}{r}+\frac{u_\theta\cot\theta}{r}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right)\right\}
\end{aligned}
\qquad (15)
$$

ただし，$\Theta=\vec{\nabla}\cdot\vec{u}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2u_r\right)+\frac{1}{r\sin\theta}\frac{\partial\left(u_\theta\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$

以下，電磁流体力学に必要となる基礎方程式（成分表示）を示す．

＜質量保存式＞

$$
\frac{\partial\rho}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\rho=-\rho\vec{\nabla}\cdot\vec{u}
$$

式(8), (9)を使って，

$$
\frac{\partial\rho}{\partial t}+u_r\frac{\partial\rho}{\partial r}+\frac{u_\theta}{r}\frac{\partial\rho}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial\rho}{\partial\phi}=-\rho\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2u_r\right)+\frac{1}{r\sin\theta}\frac{\partial\left(u_\theta\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right\}\qquad (16)
$$

＜運動方程式＞

$$
\rho\left\{\frac{\partial\vec{u}}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\right\}=-\vec{\nabla}p+\mu\nabla^2\vec{u}+\vec{j}\times\vec{b}+\rho\vec{g}
$$

式(10), (13)を使って，

$$
\begin{aligned}
&\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\theta{}^2+u_\phi{}^2}{r}\right)=-\frac{\partial p}{\partial r}+\left(j_\theta b_\phi-j_\phi b_\theta\right)+\rho g_r\\
&\quad+\mu\left(\nabla^2u_r-\frac{2u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2u_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)
\end{aligned}
\qquad (17)
$$

$$
\begin{aligned}
&\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}+\frac{u_\theta u_r-u_\phi{}^2\cot\theta}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\left(j_\phi b_r-j_r b_\phi\right)+\rho g_\theta\\
&\quad+\mu\left(\nabla^2u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right)
\end{aligned}
\qquad (18)
$$

$$
\begin{aligned}
&\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_\phi u_r}{r}+\frac{u_\phi u_\theta\cot\theta}{r}\right)=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}+\left(j_r b_\theta-j_\theta b_r\right)+\rho g_\phi\\
&\quad+\mu\left(\nabla^2u_\phi-\frac{u_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\right)
\end{aligned}
\qquad (19)
$$

＜エネルギー方程式＞

$$\frac{DT}{Dt}=\alpha\nabla^2 T+\Phi$$

式(7), (9), (15)を使って,

$$\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}=\alpha\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}\right\}$$
$$+\mu\left[\begin{array}{l}2\left(\dfrac{\partial u_r}{\partial r}\right)^2+2\left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta}+\dfrac{u_r}{r}\right)^2+2\left(\dfrac{u_r}{r}+\dfrac{u_\theta\cot\theta}{r}+\dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial\phi}\right)^2\\ +\left(\dfrac{\partial u_\theta}{\partial r}+\dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}-\dfrac{u_\theta}{r}\right)^2+\left(\dfrac{\partial u_\phi}{\partial r}+\dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial\phi}-\dfrac{u_\phi}{r}\right)^2+\left(\dfrac{1}{r}\dfrac{\partial u_\phi}{\partial\theta}+\dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial\phi}-\dfrac{u_\phi\cot\theta}{r}\right)^2\\ -\dfrac{2}{3}\left\{\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2u_r\right)+\dfrac{1}{r\sin\theta}\dfrac{\partial\left(u_\theta\sin\theta\right)}{\partial\theta}+\dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial\phi}\right\}^2\end{array}\right] \tag{20}$$

＜電荷保存則＞

$$\vec{\nabla}\cdot\vec{j}=0$$

式(8)から

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 j_r\right)+\frac{1}{r\sin\theta}\frac{\partial\left(j_\theta\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial j_\phi}{\partial\phi}=0 \tag{21}$$

＜Ohm の法則＞

$$\vec{j}=\sigma\left(-\vec{\nabla}\varphi-\frac{\partial\vec{a}}{\partial t}+\vec{u}\times\vec{b}\right)$$

式(6)から

$$\vec{j}=\underbrace{\sigma\left(-\frac{\partial\varphi}{\partial r}-\frac{\partial a_r}{\partial t}+u_\theta b_\phi-u_\phi b_\theta\right)}_{j_r}\vec{e}_r+\underbrace{\sigma\left(-\frac{1}{r}\frac{\partial\varphi}{\partial\theta}-\frac{\partial a_\theta}{\partial t}+u_\phi b_r-u_r b_\phi\right)}_{j_\theta}\vec{e}_\theta$$
$$+\underbrace{\sigma\left(-\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}-\frac{\partial a_\phi}{\partial t}+u_r b_\theta-u_\theta b_r\right)}_{j_\phi}\vec{e}_\phi \tag{22}$$

＜誘導方程式＞

$$\frac{\partial\vec{b}}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\vec{b}=\left(\vec{b}\cdot\vec{\nabla}\right)\vec{u}+\nu_m\nabla^2\vec{b}$$

式(10), (13), (14)を使って

$$\frac{\partial b_r}{\partial t}+u_r\frac{\partial b_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_r}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial b_r}{\partial\phi}=b_r\frac{\partial u_r}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_r}{\partial\theta}+\frac{b_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi}$$
$$+\nu_m\left(\nabla^2 b_r-\frac{2b_r}{r^2}-\frac{2}{r^2}\frac{\partial b_\theta}{\partial\theta}-\frac{2b_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial b_\phi}{\partial\phi}\right) \tag{23}$$

$$
\frac{\partial b_\theta}{\partial t}+u_r\frac{\partial b_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_\theta}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial b_\theta}{\partial \phi}+\frac{u_\theta b_r}{r}=b_r\frac{\partial u_\theta}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{b_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}+\frac{b_\theta u_r}{r}
$$
$$
+\nu_m\left(\nabla^2 b_\theta+\frac{2}{r^2}\frac{\partial b_r}{\partial\theta}-\frac{b_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial b_\phi}{\partial\phi}\right) \tag{24}
$$

$$
\frac{\partial b_\phi}{\partial t}+u_r\frac{\partial b_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_\phi}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial b_\phi}{\partial \phi}+\frac{u_\phi b_r}{r}+\frac{u_\phi b_\theta\cot\theta}{r}=b_r\frac{\partial u_\phi}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{b_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}
$$
$$
+\frac{b_\phi u_r}{r}+\frac{b_\phi u_\theta\cot\theta}{r}+\nu_m\left(\nabla^2 b_\phi-\frac{b_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial b_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial b_\theta}{\partial\phi}\right) \tag{25}
$$

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