偏微分方程式の記述法について

流体力学や電磁気学では，偏微分方程式がよく登場する．その中のベクトルやテンソルはアインシュタインの縮約記法に基づいて記述されることがあるので，ここで説明する．

１．ベクトル，テンソルの表記

ベクトル，テンソルはそれぞれベクトルの基底を用いて次のように表す．

$$\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z = r_1\vec{e}_1 + r_2\vec{e}_2 + r_3\vec{e}_3 = \sum_{i=1}^{3} r_i\vec{e}_i$$

$$\begin{aligned}\boldsymbol{\tau} &= \tau_{11}\vec{e}_1\otimes\vec{e}_1 + \tau_{12}\vec{e}_1\otimes\vec{e}_2 + \tau_{13}\vec{e}_1\otimes\vec{e}_3 \\ &\quad + \tau_{21}\vec{e}_2\otimes\vec{e}_1 + \tau_{22}\vec{e}_2\otimes\vec{e}_2 + \tau_{23}\vec{e}_2\otimes\vec{e}_3 \\ &\quad + \tau_{31}\vec{e}_3\otimes\vec{e}_1 + \tau_{32}\vec{e}_3\otimes\vec{e}_2 + \tau_{33}\vec{e}_3\otimes\vec{e}_3 = \sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}\vec{e}_i\otimes\vec{e}_j\end{aligned}$$

添え字表記では総和記号と基底ベクトルを省略して

$$\vec{r} = r_i$$

$$\boldsymbol{\tau} = \tau_{ij}$$

と書く．同様にナブラ演算子は次のように表す．演算子の前に基本ベクトルを置く．

$$\vec{\nabla} = \vec{e}_1\frac{\partial}{\partial x_1} + \vec{e}_2\frac{\partial}{\partial x_2} + \vec{e}_3\frac{\partial}{\partial x_3} = \sum_{i=1}^{3}\vec{e}_i\frac{\partial}{\partial x_i} \quad \Rightarrow \quad \vec{\nabla} = \frac{\partial}{\partial x_i} \equiv \partial_i$$

２．添え字演算

まず添え字演算で用いる演算子について説明する．演算子はそれぞれクロネッカーのデルタ，エディントンのイプシロンであり，次のように定義される．

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\varepsilon_{ijk} = \begin{cases} 1 & (i,j,k) = (1,2,3),(2,3,1),(3,1,2) \\ -1 & (i,j,k) = (1,3,2),(2,1,3),(3,2,1) \\ 0 & \text{else} \end{cases}$$

添え字演算では総和の規約により，例えば次式のように同じ添え字が出てきたら，その添え字について 1 から 3 までの和をとる．

$$a_i b_i = a_j b_j = a_1 b_1 + a_2 b_2 + a_3 b_3$$

以上の規則を基に添え字表記によるベクトルの演算は次のようになる．

・ベクトルの内積

$$\vec{a}\cdot\vec{b}=\left(\vec{e}_1a_1+\vec{e}_2a_2+\vec{e}_3a_3\right)\cdot\left(\vec{e}_1b_1+\vec{e}_2b_2+\vec{e}_3b_3\right)=\sum_{i=1}^{3}\vec{e}_ia_i\cdot\sum_{j=1}^{3}\vec{e}_jb_j=\sum_{i=1}^{3}\sum_{j=1}^{3}\underbrace{\vec{e}_i\cdot\vec{e}_j}_{\delta_{ij}}a_ib_j$$

$$\Rightarrow \delta_{ij}a_ib_j=a_ib_i=a_jb_j=a_1b_1+a_2b_2+a_3b_3$$

・<u>ベクトルの外積</u>

$$\vec{a}\times\vec{b}=\left(\vec{e}_1a_1+\vec{e}_2a_2+\vec{e}_3a_3\right)\times\left(\vec{e}_1b_1+\vec{e}_2b_2+\vec{e}_3b_3\right)=\sum_{i=1}^{3}\sum_{j=1}^{3}\vec{e}_i\times\vec{e}_ja_ib_j=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{kij}\vec{e}_ka_ib_j$$

$$\Rightarrow \varepsilon_{kij}\vec{e}_ka_ib_j=\vec{e}_1\left(a_2b_3-a_3b_2\right)+\vec{e}_2\left(a_3b_1-a_1b_3\right)+\vec{e}_3\left(a_1b_2-a_2b_1\right)=\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

・<u>スカラ三重積</u>

$$\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)=\sum_{i}^{3}\vec{e}_ic_i\cdot\left(\sum_{j}^{3}\vec{e}_ja_j\times\sum_{k}^{3}\vec{e}_kb_k\right)=\sum_{i}^{3}\sum_{j}^{3}\sum_{k}^{3}\vec{e}_i\cdot\left(\vec{e}_j\times\vec{e}_k\right)c_ia_jb_k$$

$$=\sum_{i}^{3}\sum_{j}^{3}\sum_{k}^{3}\vec{e}_i\cdot\left(\sum_{l}^{3}\varepsilon_{ljk}\vec{e}_l\right)c_ia_jb_k=\sum_{i}^{3}\sum_{j}^{3}\sum_{k}^{3}\sum_{l}^{3}\varepsilon_{ljk}\underbrace{\vec{e}_i\cdot\vec{e}_l}_{\delta_{il}}c_ia_jb_k=\sum_{i}^{3}\sum_{j}^{3}\sum_{k}^{3}\sum_{l}^{3}\varepsilon_{ljk}\delta_{il}c_ia_jb_k$$

$$=\sum_{i}^{3}\sum_{j}^{3}\sum_{k}^{3}\varepsilon_{ijk}c_ia_jb_k \qquad \therefore\ \ \varepsilon_{ijk}=\vec{e}_i\cdot\left(\vec{e}_j\times\vec{e}_k\right)$$

和の記号を省略して

$$\varepsilon_{ijk}c_ia_jb_k=\varepsilon_{kij}b_kc_ia_j=\vec{b}\cdot\left(\vec{c}\times\vec{a}\right)=\varepsilon_{jki}a_jb_kc_i=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$$

$$=c_1\left(a_2b_3-a_3b_2\right)+c_2\left(a_3b_1-a_1b_3\right)+c_3\left(a_1b_2-a_2b_1\right)=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

・ <u>ベクトル三重積</u>

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\varepsilon_{ijk}\left(\varepsilon_{jlm}a_lb_m\right)c_k=\varepsilon_{jki}\varepsilon_{jlm}a_lb_mc_k=\left(\delta_{kl}\delta_{im}-\delta_{km}\delta_{il}\right)a_lb_mc_k$$

$$=a_kb_ic_k-a_ib_kc_k=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{b}\cdot\vec{c}\right)\vec{a}$$

$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{c}\times\vec{b}\right)\times\vec{a}=\left(\vec{c}\cdot\vec{a}\right)\vec{b}-\left(\vec{b}\cdot\vec{a}\right)\vec{c}$　　かける順番によって結果が異なることに注意

（$\varepsilon_{ijk}\varepsilon_{ilm}=\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}$ は回転や外積を二回取るときに，使われる公式である.）

・ <u>ベクトルとテンソルの内積</u>

$$\vec{a}\cdot\boldsymbol{\tau}=\sum_{i=1}^{3}\vec{e}_i a_i\cdot\sum_{j=1}^{3}\sum_{k=1}^{3}\vec{e}_j\vec{e}_k\tau_{jk}=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\underbrace{\vec{e}_i\cdot\vec{e}_j}_{\delta_{ij}}\vec{e}_k a_i\tau_{jk}$$

$$\Rightarrow \delta_{ij}a_i\tau_{jk}=a_j\tau_{jk}=a_i\tau_{ik}=\begin{pmatrix} a_1\tau_{11}+a_2\tau_{21}+a_3\tau_{31} \\ a_1\tau_{12}+a_2\tau_{22}+a_3\tau_{32} \\ a_1\tau_{13}+a_2\tau_{23}+a_3\tau_{33} \end{pmatrix}$$

$$=\vec{e}_1\left(a_1\tau_{11}+a_2\tau_{21}+a_3\tau_{31}\right)+\vec{e}_2\left(a_1\tau_{12}+a_2\tau_{22}+a_3\tau_{32}\right)+\vec{e}_3\left(a_1\tau_{13}+a_2\tau_{23}+a_3\tau_{33}\right)$$

順序を入れ替えた場合

$$\boldsymbol{\tau}\cdot\vec{a}=\sum_{j=1}^{3}\sum_{k=1}^{3}\vec{e}_j\vec{e}_k\tau_{jk}\cdot\sum_{i=1}^{3}\vec{e}_i a_i=\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{i=1}^{3}\vec{e}_j\underbrace{\vec{e}_k\cdot\vec{e}_i}_{\delta_{ki}}\tau_{jk}a_i$$

$$\Rightarrow \delta_{ki}\tau_{jk}a_i=\tau_{ji}a_i=\tau_{jk}a_k=\begin{pmatrix} \tau_{11}a_1+\tau_{12}a_2+\tau_{13}a_3 \\ \tau_{21}a_1+\tau_{22}a_2+\tau_{23}a_3 \\ \tau_{31}a_1+\tau_{32}a_2+\tau_{33}a_3 \end{pmatrix}$$

$$=\vec{e}_1\left(\tau_{11}a_1+\tau_{12}a_2+\tau_{13}a_3\right)+\vec{e}_2\left(\tau_{21}a_1+\tau_{22}a_2+\tau_{23}a_3\right)+\vec{e}_3\left(\tau_{31}a_1+\tau_{32}a_2+\tau_{33}a_3\right)$$

順序によって結果が異なるので，結果を同じにするには転置を取る必要がある．

- テンソル同士の内積

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sum_{i=1}^{3}\sum_{j=1}^{3}\vec{e}_i\vec{e}_j\sigma_{ij}\cdot\sum_{k=1}^{3}\sum_{l=1}^{3}\vec{e}_k\vec{e}_l\tau_{kl}=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\vec{e}_i\underbrace{\vec{e}_j\cdot\vec{e}_k}_{\delta_{jk}}\vec{e}_l\sigma_{ij}\tau_{kl}=\sum_{j=1}^{3}\sum_{l=1}^{3}\delta_{jk}\vec{e}_i\vec{e}_l\sigma_{ij}\tau_{kl}$$

$$\Rightarrow \delta_{jk}\sigma_{ij}\tau_{kl}=\sigma_{ik}\tau_{kl}=\sigma_{i1}\tau_{1l}+\sigma_{i2}\tau_{2l}+\sigma_{i3}\tau_{3l}$$

$$=\vec{e}_1\vec{e}_1\left(\sigma_{11}\tau_{11}+\sigma_{12}\tau_{21}+\sigma_{13}\tau_{31}\right)+\vec{e}_1\vec{e}_2\left(\sigma_{11}\tau_{12}+\sigma_{12}\tau_{22}+\sigma_{13}\tau_{32}\right)+\vec{e}_1\vec{e}_3\left(\sigma_{11}\tau_{13}+\sigma_{12}\tau_{23}+\sigma_{13}\tau_{33}\right)$$
$$+\vec{e}_2\vec{e}_1\left(\sigma_{21}\tau_{11}+\sigma_{22}\tau_{21}+\sigma_{23}\tau_{31}\right)+\vec{e}_2\vec{e}_2\left(\sigma_{21}\tau_{12}+\sigma_{22}\tau_{22}+\sigma_{23}\tau_{32}\right)+\vec{e}_2\vec{e}_3\left(\sigma_{21}\tau_{13}+\sigma_{22}\tau_{23}+\sigma_{23}\tau_{33}\right)$$
$$+\vec{e}_3\vec{e}_1\left(\sigma_{31}\tau_{11}+\sigma_{32}\tau_{21}+\sigma_{33}\tau_{31}\right)+\vec{e}_3\vec{e}_2\left(\sigma_{31}\tau_{12}+\sigma_{32}\tau_{22}+\sigma_{33}\tau_{32}\right)+\vec{e}_3\vec{e}_3\left(\sigma_{31}\tau_{13}+\sigma_{32}\tau_{23}+\sigma_{33}\tau_{33}\right)$$

転置を考えると，4 通りの内積計算がある．

- テンソル同士の複内積

$$\boldsymbol{\sigma}:\boldsymbol{\tau}=\sum_{i=1}^{3}\sum_{j=1}^{3}\vec{e}_i\vec{e}_j\sigma_{ij}:\sum_{k=1}^{3}\sum_{l=1}^{3}\vec{e}_k\vec{e}_l\tau_{kl}=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\vec{e}_i\vec{e}_j:\vec{e}_k\vec{e}_l\sigma_{ij}\tau_{kl}=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jk}\delta_{il}\sigma_{ij}\tau_{kl}$$

$$\Rightarrow \delta_{jk}\delta_{il}\sigma_{ij}\tau_{kl}=\sigma_{ik}\tau_{ki}$$

$$=\sigma_{11}\tau_{11}+\sigma_{12}\tau_{21}+\sigma_{13}\tau_{31}+\sigma_{21}\tau_{12}+\sigma_{22}\tau_{22}+\sigma_{23}\tau_{32}+\sigma_{31}\tau_{13}+\sigma_{32}\tau_{23}+\sigma_{33}\tau_{33}$$

３．一階微分および二階微分の演算

＜1 階微分＞

スカラの勾配は,

$$\vec{\nabla}\phi \equiv \partial_i \phi = \frac{\partial \phi}{\partial x_i} = \vec{e}_1 \frac{\partial \phi}{\partial x_1} + \vec{e}_2 \frac{\partial \phi}{\partial x_2} + \vec{e}_3 \frac{\partial \phi}{\partial x_3}$$ （ベクトル）

ベクトルの勾配は,

$$\vec{\nabla}\vec{u} = \vec{\nabla} \otimes \vec{u} \equiv \partial_i u_j = \frac{\partial u_j}{\partial x_i}$$ （2 階のテンソル）

ベクトルの発散は,

$$\vec{\nabla} \cdot \vec{u} \equiv \delta_{ij} \partial_i u_j = \partial_j u_j = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$ （スカラ）

テンソルの発散は，2 種類あって,

$$\begin{aligned}\vec{\nabla} \cdot \boldsymbol{\tau} &\equiv \delta_{ij} \partial_i \tau_{jk} = \partial_j \tau_{jk} = \frac{\partial \tau_{ik}}{\partial x_i} \\ &= \vec{e}_1 \left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} \right) + \vec{e}_2 \left(\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} \right) + \vec{e}_3 \left(\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} \right)\end{aligned}$$ （ベクトル）

$$\begin{aligned}\vec{\nabla} \cdot {}^t\boldsymbol{\tau} &\equiv \delta_{ij} \partial_i \tau_{kj} = \partial_j \tau_{kj} = \frac{\partial \tau_{ki}}{\partial x_i} \\ &= \vec{e}_1 \left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} \right) + \vec{e}_2 \left(\frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} \right) + \vec{e}_3 \left(\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} \right)\end{aligned}$$ （ベクトル）

ベクトルの回転は,

$$\begin{aligned}\vec{\nabla} \times \vec{u} &\equiv \varepsilon_{ijk} \partial_j u_k = \varepsilon_{ijk} \vec{e}_i \frac{\partial u_k}{\partial x_j} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \\ &= \vec{e}_1 \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) + \vec{e}_2 \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) + \vec{e}_3 \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)\end{aligned}$$ （ベクトル）

テンソルの回転は,

$$\begin{aligned}\vec{\nabla} \times \boldsymbol{\tau} &\equiv \varepsilon_{ijk} \partial_j \tau_{kl} = \varepsilon_{ijk} \vec{e}_i \vec{e}_l \frac{\partial \tau_{kl}}{\partial x_j} \\ &= \vec{e}_1\vec{e}_1 \left(\frac{\partial \tau_{31}}{\partial x_2} - \frac{\partial \tau_{21}}{\partial x_3} \right) + \vec{e}_1\vec{e}_2 \left(\frac{\partial \tau_{32}}{\partial x_2} - \frac{\partial \tau_{22}}{\partial x_3} \right) + \vec{e}_1\vec{e}_3 \left(\frac{\partial \tau_{33}}{\partial x_2} - \frac{\partial \tau_{23}}{\partial x_3} \right) \\ &\quad + \vec{e}_2\vec{e}_1 \left(\frac{\partial \tau_{11}}{\partial x_3} - \frac{\partial \tau_{31}}{\partial x_1} \right) + \vec{e}_2\vec{e}_2 \left(\frac{\partial \tau_{12}}{\partial x_3} - \frac{\partial \tau_{32}}{\partial x_1} \right) + \vec{e}_2\vec{e}_3 \left(\frac{\partial \tau_{13}}{\partial x_3} - \frac{\partial \tau_{33}}{\partial x_1} \right) \\ &\quad + \vec{e}_3\vec{e}_1 \left(\frac{\partial \tau_{21}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2} \right) + \vec{e}_3\vec{e}_2 \left(\frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{12}}{\partial x_2} \right) + \vec{e}_3\vec{e}_3 \left(\frac{\partial \tau_{23}}{\partial x_1} - \frac{\partial \tau_{13}}{\partial x_2} \right)\end{aligned}$$ （2 階のテンソル）

この発散を取ったものはゼロベクトルになる．

＜2 階微分＞

スカラの勾配の発散は，

$$\vec{\nabla}\cdot\left(\vec{\nabla}\phi\right)\equiv\delta_{ij}\partial_i\left(\partial_j\phi\right)=\partial_j\partial_j\phi=\frac{\partial^2\phi}{\partial x_i\partial x_i}=\nabla^2\phi=\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2}\qquad\text{（スカラ）}$$

ベクトルの勾配の発散は，

$$\begin{aligned}\vec{\nabla}\cdot\left(\vec{\nabla}\vec{u}\right)&=\vec{\nabla}\cdot\left(\vec{\nabla}\otimes\vec{u}\right)\equiv\delta_{ij}\partial_i\left(\partial_j u_k\right)=\partial_j\left(\partial_j u_k\right)=\partial_i\partial_i u_k=\frac{\partial^2 u_k}{\partial x_i\partial x_i}=\nabla^2\vec{u}\\&=\vec{e}_1\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_1}{\partial x_3^2}\right)+\vec{e}_2\left(\frac{\partial^2 u_2}{\partial x_1^2}+\frac{\partial^2 u_2}{\partial x_2^2}+\frac{\partial^2 u_2}{\partial x_3^2}\right)+\vec{e}_3\left(\frac{\partial^2 u_3}{\partial x_1^2}+\frac{\partial^2 u_3}{\partial x_2^2}+\frac{\partial^2 u_3}{\partial x_3^2}\right)\end{aligned}\qquad\text{（ベクトル）}$$

スカラの勾配の回転は，

$$\begin{aligned}\vec{\nabla}\times\left(\vec{\nabla}\phi\right)&\equiv\varepsilon_{ijk}\partial_j\left(\partial_k\phi\right)=\frac{1}{2}\left\{\varepsilon_{ijk}\partial_j\left(\partial_k\phi\right)+\varepsilon_{ijk}\partial_j\left(\partial_k\phi\right)\right\}\\&=\frac{1}{2}\left\{\varepsilon_{ijk}\partial_j\left(\partial_k\phi\right)+\varepsilon_{ikj}\partial_k\left(\partial_j\phi\right)\right\}=\frac{1}{2}\left\{\varepsilon_{ijk}\partial_j\left(\partial_k\phi\right)-\varepsilon_{ijk}\partial_k\left(\partial_j\phi\right)\right\}\\&=\frac{\varepsilon_{ijk}}{2}\underbrace{\left\{\partial_j\left(\partial_k\phi\right)-\partial_k\left(\partial_j\phi\right)\right\}}_{0_{jk}}=0_i=\vec{0}\end{aligned}\qquad\text{（ゼロベクトル）}$$

ベクトルの発散の勾配は，

$$\begin{aligned}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right)&\equiv\partial_k\left(\delta_{ij}\partial_i u_j\right)=\partial_k\left(\partial_j u_j\right)=\partial_k\left(\partial_i u_i\right)\\&=\vec{e}_1\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)+\vec{e}_2\frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\\&\quad+\vec{e}_3\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\end{aligned}\qquad\text{（ベクトル）}$$

テンソルの発散の発散は，どちらも結果は同じである．

$$\begin{aligned}\vec{\nabla}\cdot\left(\vec{\nabla}\cdot\boldsymbol{\tau}\right)&\equiv\delta_{lk}\partial_l\left(\delta_{ij}\partial_i\tau_{jk}\right)=\partial_l\left(\delta_{ij}\partial_i\tau_{jl}\right)=\partial_l\left(\partial_j\tau_{jl}\right)=\partial_k\left(\partial_i\tau_{ik}\right)=\frac{\partial^2\tau_{ik}}{\partial x_i\partial x_k}\\&=\frac{\partial^2\tau_{11}}{\partial x_1\partial x_1}+\frac{\partial^2\tau_{12}}{\partial x_1\partial x_2}+\frac{\partial^2\tau_{13}}{\partial x_1\partial x_3}+\frac{\partial^2\tau_{21}}{\partial x_2\partial x_1}+\frac{\partial^2\tau_{22}}{\partial x_2\partial x_2}+\frac{\partial^2\tau_{23}}{\partial x_2\partial x_3}+\frac{\partial^2\tau_{31}}{\partial x_3\partial x_1}+\frac{\partial^2\tau_{32}}{\partial x_3\partial x_2}+\frac{\partial^2\tau_{33}}{\partial x_3\partial x_3}\end{aligned}$$

$$\begin{aligned}\vec{\nabla}\cdot\left(\vec{\nabla}\cdot{}^t\boldsymbol{\tau}\right)&=\delta_{lk}\partial_l\left(\delta_{ij}\partial_i\tau_{kj}\right)=\partial_k\left(\delta_{ij}\partial_i\tau_{kj}\right)=\partial_k\left(\partial_j\tau_{kj}\right)=\frac{\partial^2\tau_{kj}}{\partial x_j\partial x_k}\\&=\frac{\partial^2\tau_{11}}{\partial x_1\partial x_1}+\frac{\partial^2\tau_{21}}{\partial x_1\partial x_2}+\frac{\partial^2\tau_{31}}{\partial x_1\partial x_3}+\frac{\partial^2\tau_{12}}{\partial x_2\partial x_1}+\frac{\partial^2\tau_{22}}{\partial x_2\partial x_2}+\frac{\partial^2\tau_{32}}{\partial x_2\partial x_3}+\frac{\partial^2\tau_{13}}{\partial x_3\partial x_1}+\frac{\partial^2\tau_{23}}{\partial x_3\partial x_2}+\frac{\partial^2\tau_{33}}{\partial x_3\partial x_3}\end{aligned}$$

ベクトルの回転の発散は，

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{u}\right)\equiv\delta_{li}\partial_l\left(\varepsilon_{ijk}\partial_j u_k\right)=\partial_i\left(\varepsilon_{ijk}\partial_j u_k\right)=\varepsilon_{ijk}\partial_i\partial_j u_k$$
$$=\frac{1}{2}\left(\varepsilon_{ijk}\partial_i\partial_j u_k+\varepsilon_{ijk}\partial_i\partial_j u_k\right)=\frac{1}{2}\left(\varepsilon_{ijk}\partial_i\partial_j u_k+\varepsilon_{jik}\partial_j\partial_i u_k\right)\qquad\text{(ゼロ)}$$
$$=\frac{1}{2}\left(\varepsilon_{ijk}\partial_i\partial_j u_k-\varepsilon_{ijk}\partial_j\partial_i u_k\right)=\frac{\varepsilon_{ijk}}{2}\underbrace{\left(\partial_i\partial_j u_k-\partial_j\partial_i u_k\right)}_{0_{ijk}}=0$$

ベクトルの回転の回転は，

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{u}\right)\equiv\varepsilon_{lmi}\partial_m\left(\varepsilon_{ijk}\partial_j u_k\right)=\varepsilon_{ijk}\varepsilon_{ilm}\frac{\partial^2 u_k}{\partial x_j\partial x_m}=\left(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}\right)\frac{\partial^2 u_k}{\partial x_j\partial x_m}$$
$$=\frac{\partial^2 u_k}{\partial x_l\partial x_k}-\frac{\partial^2 u_l}{\partial x_m\partial x_m}=\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)-\frac{\partial^2 u_i}{\partial x_k\partial x_k}=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right)-\nabla^2\vec{u}\qquad\text{(ベクトル)}$$

$\varepsilon_{ijk}\varepsilon_{ilm}=\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}$ は回転や外積を二回取るときに，使われる公式である．

要約すれば，以下のように記憶しておくと役立つだろう．

勾配：$\dfrac{\partial b_j}{\partial x_i}=\partial_i b_j$ （テンソル）

発散：$\delta_{ij}\dfrac{\partial b_j}{\partial x_i}=\dfrac{\partial b_j}{\partial x_j}=\partial_j b_j$ （スカラ）

回転：$\varepsilon_{ijk}\dfrac{\partial b_k}{\partial x_j}=\varepsilon_{ijk}\partial_j b_k$ （ベクトル）

$\vec{e}_i\cdot\vec{e}_j=\delta_{ij}$

$\vec{e}_i\cdot\left(\vec{e}_j\times\vec{e}_k\right)=\varepsilon_{ijk}$

$\varepsilon_{ijk}\varepsilon_{ilm}=\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}$

4．添え字演算から諸公式の導出

$$\vec{\nabla}\cdot\left(f\,\vec{u}\right)=\delta_{ij}\partial_i\left(fu_j\right)=\partial_j\left(fu_j\right)=u_j\partial_j f+f\partial_j u_j=\vec{u}\cdot\vec{\nabla}f+f\vec{\nabla}\cdot\vec{u}$$

$$\vec{\nabla}\cdot\left(\vec{u}\times\vec{w}\right)=\delta_{li}\partial_l\left(\varepsilon_{ijk}u_j w_k\right)=\partial_i\left(\varepsilon_{ijk}u_j w_k\right)$$
$$=w_k\partial_i\varepsilon_{ijk}u_j+u_j\partial_i\varepsilon_{ijk}w_k=w_k\varepsilon_{kij}\partial_i u_j-u_j\varepsilon_{jik}\partial_i w_k=\vec{w}\cdot\left(\vec{\nabla}\times\vec{u}\right)-\vec{u}\cdot\left(\vec{\nabla}\times\vec{w}\right)$$

$$\vec{\nabla}\times\left(f\,\vec{u}\right)=\varepsilon_{ijk}\partial_j\left(fu_k\right)=u_k\varepsilon_{ijk}\partial_j f+f\varepsilon_{ijk}\partial_j u_k$$
$$=\varepsilon_{ijk}\frac{\partial f}{\partial x_j}u_k+f\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}=\vec{\nabla}f\times\vec{u}+f\vec{\nabla}\times\vec{u}$$

$$\vec{\nabla}\times\left(\vec{u}\times\vec{w}\right)=\varepsilon_{ijk}\partial_j\left(\varepsilon_{klm}u_l w_m\right)=\varepsilon_{kij}\varepsilon_{klm}\partial_j\left(u_l w_m\right)=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\left(w_m\partial_j u_l+u_l\partial_j w_m\right)$$
$$=\delta_{il}\left(w_j\partial_j u_l+u_l\partial_j w_j\right)-\delta_{jl}\left(w_i\partial_j u_l+u_l\partial_j w_i\right)=\left(w_j\partial_j u_i+u_i\partial_j w_j\right)-\left(w_i\partial_j u_j+u_j\partial_j w_i\right)$$
$$=w_j\partial_j u_i-u_j\partial_j w_i+u_i\partial_j w_j-w_i\partial_j u_j=\left(\vec{w}\cdot\vec{\nabla}\right)\vec{u}-\left(\vec{u}\cdot\vec{\nabla}\right)\vec{w}+\vec{u}\left(\vec{\nabla}\cdot\vec{w}\right)-\vec{w}\left(\vec{\nabla}\cdot\vec{u}\right)$$

$$\vec{\nabla}\left(fg\right)=\partial_i\left(fg\right)=g\partial_i f+f\partial_i g=g\vec{\nabla}f+f\vec{\nabla}g$$

$$\vec{\nabla}\left(\vec{u}\cdot\vec{w}\right)=\partial_i\left(u_k w_k\right)=w_k\partial_i u_k+u_k\partial_i w_k$$
$$=w_k\left(\partial_i u_k-\partial_k u_i\right)+w_k\partial_k u_i+u_k\left(\partial_i w_k-\partial_k w_i\right)+u_k\partial_k w_i$$
$$=\vec{w}\times\left(\vec{\nabla}\times\vec{u}\right)+\left(\vec{w}\cdot\vec{\nabla}\right)\vec{u}+\vec{u}\times\left(\vec{\nabla}\times\vec{w}\right)+\left(\vec{u}\cdot\vec{\nabla}\right)\vec{w}$$
$$\left(\begin{aligned}\because\ \ &\vec{A}\times\left(\vec{\nabla}\times\vec{B}\right)=\varepsilon_{ijk}A_j\left(\varepsilon_{klm}\partial_l B_m\right)=\varepsilon_{ijk}\varepsilon_{klm}A_j\partial_l B_m=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{lj}\right)A_j\partial_l B_m\\ &=\delta_{il}\delta_{jm}A_j\partial_l B_m-\delta_{im}\delta_{lj}A_j\partial_l B_m=A_j\partial_i B_j-A_j\partial_j B_i=A_j\left(\partial_i B_j-\partial_j B_i\right)\end{aligned}\right)$$

最後の式において，特に$\vec{u}=\vec{w}$とすると，次の関係式が得られる．

$$\vec{\nabla}\left(\vec{u}\cdot\vec{u}\right)=2\vec{u}\times\underbrace{\left(\vec{\nabla}\times\vec{u}\right)}_{\vec{\omega}}+2\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\quad\Rightarrow\quad\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}=\vec{\nabla}\left(\frac{\vec{u}\cdot\vec{u}}{2}\right)-\vec{u}\times\vec{\omega}$$

この関係式は，Navier-Stokes 方程式の回転を取るときに役立つ．その結果，下記のような渦度輸送方程式を得る．

$$\frac{\partial\vec{\omega}}{\partial t}=\vec{\nabla}\times\left(\vec{u}\times\vec{\omega}\right)+\nu\nabla^2\vec{\omega}$$

また，電磁流体力学などで現れる Lorentz 力に対して，Ampère の法則を適用し，上の公式をあてはめれば，

$$\vec{f}_{em}=\vec{j}\times\vec{B}=\frac{1}{\mu_m}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}=\frac{1}{\mu_m}\left[\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}-\frac{1}{2}\vec{\nabla}\left(\vec{B}\cdot\vec{B}\right)\right]$$

となる．誘導方程式は次式で与えられる．

$$\frac{\partial\vec{B}}{\partial t}=\vec{\nabla}\times\left(\vec{u}\times\vec{B}\right)+\nu_m\nabla^2\vec{B}$$

５．偏微分方程式の表記例

例えば，質量保存式，運動方程式，およびエネルギー方程式が黒字でそれぞれ以下のように表されるとするとき，添字指標表記ではそれぞれ以下の青字のように表される．

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\left(\rho\vec{u}\right)=0\quad\Rightarrow\quad\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}\left(\rho u_j\right)=0$$

$$\rho\left(\frac{\partial \vec{u}}{\partial t}+\vec{u}\cdot\vec{\nabla}\vec{u}\right)=-\vec{\nabla}p+\mu\nabla^2\vec{u}+\vec{j}\times\vec{B}+\vec{f}$$

$$\Rightarrow\quad \rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j\partial x_j}+\varepsilon_{ijk}j_jB_k+f_i$$

$$\rho c_p\left(\frac{\partial T}{\partial t}+\vec{u}\cdot\vec{\nabla}T\right)=\vec{\nabla}\cdot\left(k\vec{\nabla}T\right)+\beta T\frac{Dp}{Dt}+\mu\Phi$$

$$\Rightarrow\quad \rho c_p\left(\frac{\partial T}{\partial t}+u_j\frac{\partial T}{\partial x_j}\right)=\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right)+\beta T\frac{Dp}{Dt}+\mu\Phi$$

$$\frac{\partial \vec{B}}{\partial t}=\vec{\nabla}\times\left(\vec{u}\times\vec{B}\right)+\nu_m\nabla^2\vec{B}$$

$$\Rightarrow\quad \frac{\partial B_i}{\partial t}=\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\varepsilon_{klm}u_lB_m\right)+\nu_m\frac{\partial^2 B_i}{\partial x_j\partial x_j}$$

$$\Rightarrow\quad \frac{\partial B_i}{\partial t}=\frac{\partial}{\partial x_j}\left(u_iB_j-u_jB_i\right)+\nu_m\frac{\partial^2 B_i}{\partial x_j\partial x_j}$$

$$\frac{\partial \vec{A}}{\partial t}=-\vec{\nabla}\phi_e+\vec{u}\times\left(\vec{\nabla}\times\vec{A}\right)+\nu_m\nabla^2\vec{A}$$

$$\Rightarrow\quad \frac{\partial A_i}{\partial t}+u_j\frac{\partial A_i}{\partial x_j}=-\frac{\partial \phi_e}{\partial x_i}+u_j\frac{\partial A_j}{\partial x_i}+\nu_m\frac{\partial^2 A_i}{\partial x_j\partial x_j}$$

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